CHARACTERS OF GRADED PARAFERMION CONFORMAL FIELD THEORY

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ABSTRACT. The graded parafermion conformal field theory at level k is a close cousin of the much-studied $\mathbb{Z}_k$ parafermion model. Three character formulas for the graded parafermion theory are presented, one bosonic, one fermionic (both previously known) and one of spinon type (which is new). The main result of this paper is a proof of the equivalence of these three forms using q-series methods combined with the combinatorics of lattice paths. The pivotal step in our approach is the observation that the graded parafermion theory — which is equivalent to the coset $\widehat{\mathrm{osp}}(1,2)_k/\widehat{\mathrm{u}}(1)$ — can be factored as $(\widehat{\mathrm{osp}}(1,2)_k/\widehat{\mathrm{su}}(2)_k) \times (\widehat{\mathrm{su}}(2)_k/\widehat{\mathrm{u}}(1))$, with the two cosets on the right equivalent to the minimal model $\mathcal{M}(k+2, 2k+3)$ and the $\mathbb{Z}_k$ parafermion model, respectively. This factorisation allows for a new combinatorial description of the graded parafermion characters in terms of the one-dimensional configuration sums of the $(k+1)$-state Andrews–Baxter–Forrester model.

1. INTRODUCTION

A conformal field theory typically has many different formulations. For instance, it may be formulated (i) as the representation theory of some extended conformal algebra, (ii) as a coset model, (iii) as a free-field representation, (iv) in terms of a quasi-particle description. Moreover, within these different categories there can be more than one description, e.g., the model under study may be described in terms of an irreducible representation of more than one conformal algebra, or have more than one coset representation. Different constructions of the same irreducible modules may lead to formally equivalent characters which structurally appear rather distinct. This poses the problem of establishing their direct equivalence. When successful, such a verification, in turn, places the physical argument underlying the construction of the characters on a sound basis.

The aim of this work is to provide direct proofs — at the level of character representations — of different algebraic descriptions of graded parafermion theories. Since we rely heavily on the non-graded parafermion theory, we will first summarize current understanding of this more familiar and much better understood conformal field theory.

1.1. **Ordinary parafermion characters.** The $\mathbb{Z}_k$ parafermion conformal theory [34] provides a good illustration of a conformal field theory with a multitude of formulations. The theory is defined by the algebra of parafermionic fields ψ_1 and $\psi_1^\dagger$ of dimension $1 - 1/k$ and central charge $2(k-1)/(k+2)$. The highest-weight modules are parametrized by an integer (Dynkin label) ℓ with $0 \le \ell < k$. We are interested in modules with fixed relative charge $2r$, where r counts the number of ψ_1

modes minus the number of $\psi_1^\dagger$ modes. Suppressing the level k, the corresponding characters will be denoted by $\hat{\chi}_{\ell,r}(q)$. Here the 'hat' has been used to distinguish ordinary and graded parafermion characters.

The $\mathbb{Z}_k$ parafermion model is known to be equivalent to the coset

$$(1.1) \qquad \frac{\widehat{\mathrm{su}}(2)_k}{\widehat{\mathrm{u}}(1)}.$$

The first derivation of an explicit expression for the parafermionic characters by Kac and Petersen relied on (1.1) and identifies the characters with the coset branching functions.

Let $\hat{\chi}_\ell(x;q)$ denote the character of the $\widehat{\mathrm{su}}(2)_k$ integrable module of Dynkin label ℓ (with $0 \le \ell \le k$), and let $K_m(x;q)$ denote $\widehat{\mathrm{u}}(1)$ character of charge m, associated to the current algebra of a boson compactified on an integer square-radius $R^2 = 2k$ [8, Sec. 14.4.4]. Then the branching functions associated to the coset (1.1) are given by the decomposition

$$(1.2) \qquad \hat{\chi}_\ell(x;q) = \sum_{m=1-k}^{k} \hat{b}_{\ell,m}(q)\, K_m(x;q).$$

The above-mentioned identification of branching functions and parafermion characters implies that for $m - \ell$ even

$$(1.3) \qquad \hat{b}_{\ell,m}(q) = \hat{\chi}_{\ell,\frac{m-\ell}{2}}(q).$$

(It can in fact easily be shown that $\hat{b}_{\ell,m}(q) = 0$ when $m - \ell$ is odd.)

According to the Weyl–Kac formula, the $\widehat{\mathrm{su}}(2)_k$ characters are given by a ratio of differences of theta functions as

$$(1.4) \qquad \hat{\chi}_\ell(x;q) = \frac{\Theta_{\ell+1}^{(k+2)}(x;q) - \Theta_{-\ell-1}^{(k+2)}(x;q)}{\Theta_1^{(2)}(x;q) - \Theta_{-1}^{(2)}(x;q)}.$$

Using (1.2), (1.3) and (1.4), Kac and Petersen [21] obtained the following expression for the parafermion characters:

$$(1.5)$$

$$\hat{\chi}_{\ell,r}^{\mathrm{B}}(q) = \frac{1}{\eta^2(q)} \Bigg\{ \left(\sum_{\substack{i\ge 0 \\ j\ge 0}} - \sum_{\substack{i<0 \\ j<0}} \right) (-1)^i q^{\left(\ell+1+(i+2j)(k+2)\right)^2/(4(k+2)) - (m+ik)^2/(4k)}$$

$$- \left(\sum_{\substack{i\ge 0 \\ j>0}} - \sum_{\substack{i<0 \\ j\le 0}} \right) (-1)^i q^{\left(\ell+1-(i+2j)(k+2)\right)^2/(4(k+2)) - (m+ik)^2/(4k)} \Bigg\}.$$

Here $2r = m - \ell$ and η is the Dedekin eta function: $\eta(q) = q^{1/24}(q;q)_\infty$, with $(q;q)_n = \prod_{i=1}^{n}(1 - q^i)$. The superscript B attached to $\hat{\chi}_{\ell,r}(q)$ indicates a bosonic or inclusion–exclusion form — a characteristic inherited from the form (1.4) for the $\widehat{\mathrm{su}}(2)_k$ characters.

Other derivations of $\hat{\chi}_{\ell,r}(q)$ have been given in the literature using a free-field representation (one boson and a pair of ghosts [9] or three bosons [27]) and the BRST construction. This has led to bosonic expressions for the characters that are only superficially different from (1.5), and can easily be proved to be equivalent by simple manipulations.

Shortly after the Kac–Petersen derivation, an intrinsically different formula for the parafermion characters was obtained. This is the famous Lepowsky–Primc expression [23] which yields the characters as a manifestly positive multiple-series or fermionic (F) form (see also [19]). Specifically,

$$(1.6) \qquad \hat{\chi}^{\mathrm{F}}_{\ell,r}(q) = q^{\Delta_\ell} \sum_{\substack{n_1,\ldots,n_{k-1}=0 \\ \frac{r+\ell}{k}+(C^{-1}n)_1\in\mathbb{Z}}}^{\infty} \frac{q^{nC^{-1}(n-e_\ell)}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}},$$

where e_i denotes the ith standard unit vector in $\mathbb{Z}^{k-1}$ (with $e_0 = e_k$ the zero vector), $n = (n_1,\ldots,n_{k-1})$ and C denotes the A_{k-1} Cartan matrix. (For more precise definitions of the employed notation, see Section 2.4.) The exponent Δ_ℓ in the above is given by

$$(1.7) \qquad \Delta_\ell = -\frac{\ell^2}{4k} + \frac{(\ell+1)^2}{4(k+2)} - \frac{1}{12}.$$

Yet a third expression for the parafermion characters can be obtained from the so-called $\widehat{\mathrm{su}}(2)_k$ spinon formula for $\hat{\chi}_\ell(q)$ which was conjectured in [6] and proved in [3, 25, 26]. Using (1.2) to extract the branching function from the spinon formula yields

(1.8)

$$\hat{\chi}^{\mathrm{S}}_{\ell,r}(q) = (q;q)_\infty \, q^{\Delta_\ell} \sum_{\substack{L,n_1,\cdots,n_{k-1}=0 \\ \frac{L+r+\ell}{k}+(C^{-1}n)_1\in\mathbb{Z}}}^{\infty} \frac{q^{L(L+\ell+2r)/k+nC^{-1}(n-e_\ell)}}{(q;q)_L(q;q)_{L+\ell+2r}} \prod_{i=1}^{k-1} \begin{bmatrix} m_i + n_i \\ n_i \end{bmatrix},$$

where $\begin{bmatrix} n \\ m \end{bmatrix}$ is a q-binomial coefficient, and where the integers m_i can be computed from L and the n_i by

$$m = C^{-1}\Big[(2L+r)e_1 + e_\ell - 2n\Big],$$

with $m = (m_1,\ldots,m_{k-1})$.

Finally, a fourth form of the parafermionic characters has been obtained in [18], relying on what can be called 'parafermionic representation theory.' The basic building block is the character of the parafermionic Verma module of relative charge $2t$, given by

$$(1.9) \qquad \hat{V}_t(q) = \sum_{j=0}^{\infty} \frac{q^{j+t}}{(q;q)_j(q;q)_{j+t}}.$$

Combined with knowledge of the explicit structure of the singular vectors in the theory, the Verma character (1.9) leads to a second bosonic form of the characters as follows:

$$(1.10) \quad \hat{\chi}^{\mathrm{B'}}_{\ell,r}(q) = q^{-\frac{1}{12}-\frac{(\ell+2r)^2}{4k}} \sum_{j=-\infty}^{\infty} q^{(k+2)\left(j+\frac{\ell+1}{2(k+2)}\right)^2}$$

$$\times \left\{ \hat{V}_{r-(k+2)j}(q) - \hat{V}_{r+\ell+1+(k+2)j}(q) \right\}.$$

Viewed as formal q-series, the identities

$$(1.11) \qquad \hat{\chi}^{\mathrm{B}}_{\ell,r}(q) = \hat{\chi}^{\mathrm{S}}_{\ell,r}(q) = \hat{\chi}^{\mathrm{F}}_{\ell,r}(q)$$

have all been demonstrated in [29]. To complete the formal proof of the equality of all four parafermion expressions it remains to be shown that, for example,

$$(1.12) \qquad \hat{\chi}^{\mathrm{B}}_{\ell,r}(q) = \hat{\chi}^{\mathrm{B'}}_{\ell,r}(q).$$

We fill this small gap in the literature in the appendix.

   This concludes our review of character representations for ordinary parafermions, and next we turn our attention to graded parafermions.

1.2. **Graded parafermion characters.** The present work is concerned with a demonstration — at the level of q-series — of the graded analogue of (1.11), that is,

$$(1.13) \qquad \chi^{\mathrm{B}}_{\ell,r}(q) = \chi^{\mathrm{S}}_{\ell,r}(q) = \chi^{\mathrm{F}}_{\ell,r}(q).$$

Here $\chi_{\ell,r}(q)$ is a graded parafermion characters of relative charge r and Dynkin label ℓ, to be introduced below. B, S and F again refer to bosonic, spinon and fermionic representations, each of which arises from a different algebraic formulation of the graded parafermion theory.

   Graded parafermions were first introduced in [7]. The underlying algebra is generated by 'fermionic' parafermions $\psi_{1/2}$ and $\psi^{\dagger}_{1/2}$ of dimension $1 - 1/(4k)$ and central charge

$$(1.14) \qquad c = -\frac{3}{2k+3}.$$

The spectrum of the model has been determined in [7], and was shown to be equivalent to that of the coset

$$(1.15) \qquad \frac{\widehat{\mathrm{osp}}(1,2)_k}{\widehat{\mathrm{u}}(1)},$$

where $\widehat{\mathrm{osp}}(1,2)_k$ is the affine extension of the Lie superalgebra osp(1,2). The latter is a graded version of su(2) obtained by adding to the usual su(2) generators $J_{\pm}$ and J_0, the two fermionic generators $F_{\pm}$ such that $F^2_{\pm} = J_{\pm}$.

   The equivalence between the graded parafermion theory and the coset (1.15) leads to the graded analogue of (1.2) and (1.3):

$$(1.16) \qquad \chi_{\ell}(x;q) = \sum_{m=1-k}^{k} \chi_{\ell,m-\ell}(q) K_m(x;q),$$

where $\chi_{\ell}(x;q)$ denotes the character of the $\widehat{\mathrm{osp}}(1,2)_k$ integrable module of Dynkin label ℓ (with $0 \le \ell \le k$), and $K_m(x;q)$ again denotes the $\widehat{\mathrm{u}}(1)$ character of charge m.

   A further analysis of the graded parafermion model has been presented in [20], resulting in an exact expression of the singular vectors and a description — in terms of jagged partitions — of the special nature of the spanning set of states. The resulting expression for the Verma character of relative charge t is given by

$$(1.17) \qquad V_t(q) = \frac{1}{(q;q)_{\infty}} \sum_{i,j=0}^{\infty} \frac{q^{\binom{2j-2i-t}{2}+j}}{(q;q)_i (q;q)_j}.$$

The associated character for the irreducible modules is an alternating sum over these Verma characters — encoding the subtraction and addition of the embedded

singular vectors — and reads [4]

$$(1.18) \quad \chi^{\mathrm{B}}_{\ell,r}(q) = q^{-\frac{(\ell+r)^2}{4k}} \sum_{j=-\infty}^{\infty} q^{\frac{1}{2}(2k+3)\left(j+\frac{2\ell+1}{2(2k+3)}\right)^2}$$

$$\times \left\{ V_{r-(2k+3)j}(q) - V_{r+2\ell+1+(2k+3)j}(q) \right\}.$$

This is the bosonic formula for graded parafermions, one of the three central object of our work. Its underlying algebraic formulation is what could be dubbed the 'graded parafermionic representation theory.'

In Section 2.3 we present an alternative, purely analytic derivation of (1.18) along the lines of the Kac–Peterson derivation of (1.5), utilizing the branching rule (1.16) and the Weyl–Kac formula for the $\widehat{\mathrm{osp}}(1,2)_k$ characters. Unlike the non-graded theory, we have chosen to only work with a single bosonic representation, dispensing of the need for B'.

The fermionic expression of a character reflects the construction of the corresponding module using a quasi-particle basis by a filling process subject to a generalized exclusion principle. When the model is viewed directly from the point of view of the parafermionic algebra, the modes of the parafermion $\psi_{1/2}$ are the natural choice for these quasi-particles. A descendant can thus be represented by the ordered sequence of its modes, and up to an overall sign this sequence forms a jagged partition [20]. The generalized exclusion principle takes the form of certain (k-dependent) difference conditions on parts of the jagged partitions. The generating function of the restricted jagged partitions yields, up to a minor adjustment, the fermionic expression for the characters of the irreducible modules:

$$(1.19) \quad \chi^{\mathrm{F}}_{\ell,r}(q) = q^{h_\ell - c/24}$$

$$\times \sum_{\substack{n_0,\ldots,n_{k-1}=0 \\ \frac{r+2\ell+n_0}{2k}+(C^{-1}n)_1 \in \mathbb{Z}}}^{\infty} \frac{q^{-n_0(n_0+2\ell)/(4k)+\binom{n_0+1}{2}+nC^{-1}(n-n_0 e_1 - e_\ell)}}{(q;q)_{n_0} \cdots (q;q)_{n_{k-1}}}.$$

Here the exponent h_ℓ is given by the $r=0$ instance of (2.1) of Section 2.1.

Finally, we describe a third form for graded parafermion characters, referred to as a spinon formula. It is inherited from the following $\widehat{\mathrm{osp}}(1,2)_k$ analogue of the $\widehat{\mathrm{su}}(2)_k$ spinon formula:

$$(1.20) \quad \chi_\ell(x;q) = q^{h_\ell - c/24 + \ell^2/(4k) - 1/24} \sum_{\substack{L_+,L_-,n_0,\cdots,n_{k-1}=0 \\ \frac{L_++L_-+n_0+\ell}{2k}+(C^{-1}n)_1 \in \mathbb{Z}}}^{\infty} x^{-(L_+-L_-)/2}$$

$$\times \frac{q^{[(L_++L_-)^2-(n_0+\ell)^2]/(4k)+\binom{n_0+1}{2}+nC^{-1}(n-n_0 e_1 - e_\ell)}}{(q;q)_{L_+}(q;q)_{L_-}(q;q)_{n_0}} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix},$$

where

$$m = C^{-1}\Big[(L_+ + L_- + n_0)e_1 + e_\ell - 2n\Big].$$

The formula (1.20) for the $\widehat{\mathrm{osp}}(1,2)_k$ character was initially conjectured using the analogy with the non-graded theory. In the non-graded context, the term spinon refers to the two components of the basic $\ell = 1$ $\widehat{\mathrm{su}}(2)_k$ WZW primary field. In

the present case however, the doublet is traded by a triplet, the third component manifesting itself in the presence of the n_0 mode in (1.20). Its derivation from a spinon-type basis of states remains to be worked out.

The expression for the parafermionic character that results from (1.20) and (1.16) is, with $L_+ = L + r + \ell$ and $L_- = L$,

(1.21)

$$\chi^{\mathrm{S}}_{\ell,r}(q) = (q;q)_\infty \, q^{h_\ell - c/24} \sum_{\substack{L,n_0,\cdots,n_{k-1}=0 \\ \frac{2L+r+2\ell+n_0}{2k} + (C^{-1}n)_1 \in \mathbb{Z}}}^{\infty} q^{L(L+r+\ell)/k - n_0(n_0+2\ell)/(4k) + \binom{n_0+1}{2}}$$

$$\times \frac{q^{nC^{-1}(n-n_0 e_1 - e_\ell)}}{(q;q)_L (q;q)_{L+r+\ell} (q;q)_{n_0}} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix},$$

where

$$m = C^{-1}\Big[(2L + r + \ell + n_0)e_1 + e_\ell - 2n\Big].$$

Our proof of (1.21) in Section 3 of course also confirms the validity of (1.20).

1.3. **Outline of the proof of** (1.13). The proof of the equivalence of the three character representations (1.18), (1.19) and (1.21) is presented in Section 3, and forms the core of this paper. Due to its very technical nature, the proof may — certainly at first reading — not be very insightful. It thus seems important to point out the main ideas and underlying physics.

Given our much better understanding of the non-graded parafermions, an obvious approach to establishing (1.13) is to connect with the non-graded character theory. To this end we use a little trick and write the coset (1.15) as

(1.22)
$$\frac{\widehat{\mathrm{osp}}(1,2)_k}{\widehat{\mathrm{u}}(1)} = \frac{\widehat{\mathrm{osp}}(1,2)_k}{\widehat{\mathrm{su}}(2)_k} \times \frac{\widehat{\mathrm{su}}(2)_k}{\widehat{\mathrm{u}}(1)},$$

where the second coset on the right is that of ordinary parafermions, see (1.1). In [11, 22] it was shown that

(1.23)
$$\frac{\widehat{\mathrm{osp}}(1,2)_k}{\widehat{\mathrm{su}}(2)_k} \simeq \mathcal{M}(k+2, 2k+3),$$

where the right-hand side denotes the non-unitary minimal model of central charge

$$c^{(k+2,2k+3)} = 1 - \frac{6(k+1)^2}{(k+2)(2k+3)}.$$

(The reader may readily check that $-\frac{3}{2k+3} = c^{(k+2,2k+3)} + \frac{2(k-1)}{k+2}$.)

Now let $\tilde{\chi}^{(p,p')}_{r,s}(q)$ be the usual $\mathcal{M}(p,p')$ Virasoro character [8]

(1.24)
$$\tilde{\chi}^{(p,p')}_{r,s}(q) = \frac{1}{\eta(q)} \sum_{\sigma \in \{\pm 1\}} \sum_{j=-\infty}^{\infty} \sigma q^{pp'\left(j + \frac{p'r - \sigma ps}{2pp'}\right)^2}.$$

Then, by (1.23) and

$$\widehat{\mathrm{osp}}(1,2)_k = \frac{\widehat{\mathrm{osp}}(1,2)_k}{\widehat{\mathrm{su}}(2)_k} \times \widehat{\mathrm{su}}(2)_k,$$

we have

$$(1.25) \qquad \chi_\ell(x;q) = \sum_{i=0}^{k} \tilde{\chi}_{i+1,2\ell+1}^{(k+2,2k+3)}(q)\hat{\chi}_i(x;q).$$

By (1.2), (1.3) (plus $\hat{b}_{\ell,m}(q) = 0$ when $m - \ell$ is odd) and (1.16) it thus follows that the decomposition (1.22) gives rise to

$$(1.26) \qquad \chi_{\ell,r-\ell}(q) = \sum_{\substack{i=0 \\ i\equiv r \pmod 2}}^{k} \tilde{\chi}_{i+1,2\ell+1}^{(k+2,2k+3)}(q)\hat{\chi}_{i,\frac{r-i}{2}}(q).$$

It is this formula that provides the necessary handle on (1.13).

Indeed, both the Virasoro and (ordinary) parafermion character on the right admit an interpretation in terms of the same combinatorial objects: the one-dimensional configuration sums $X_{r,s}^{(p,p')}(L,b;q)$ of the Andrews–Baxter–Forrester models. This leads to (see Section 3.2)

$$(1.27) \quad \chi_{\ell,r-\ell}(q) = q^{h_\ell - (r-\ell)(r+\ell)/(4k) - c/24}$$

$$\times (q;q)_\infty \sum_{\substack{n=0 \\ n\equiv r+\ell \pmod 2}}^{\infty} \frac{q^{n^2/2}}{(q;q)_n} \sum_{L=0}^{\infty} \frac{q^{(2L+r-\ell)/2}}{(q;q)_L(q;q)_{L+r}}$$

$$\times \sum_{\substack{i=0 \\ i\equiv r \pmod 2}}^{k} X_{1,i+1}^{(1,k+2)}(n,\ell+1;q^{-1})X_{1,i+1}^{(1,k+2)}(2L+r,1;q).$$

Our proof of (1.13) basically amounts to showing that (1.27) is compatible with both (1.18) and (1.21). In the case of (1.21) this requires a combinatorial (subtractionless or fermionic) method for computing the sum over products of one-dimensional configuration sums, whereas in the case of (1.18) our approach will be analytic in nature. Once the compatibility of (1.27) with (1.18) and (1.21) is established, we have achieved the desired equality $\chi_{\ell,r}^{\mathrm{B}}(q) = \chi_{\ell,r}^{\mathrm{S}}(q)$.

Showing the remaining

$$\chi_{\ell,r}^{\mathrm{S}}(q) = \chi_{\ell,r}^{\mathrm{F}}(q)$$

does not rely on (1.27), but closely follows the approach of [29] for proving

$$\hat{\chi}_{\ell,r}^{\mathrm{S}}(q) = \hat{\chi}_{\ell,r}^{\mathrm{F}}(q)$$

in the case of non-graded parafermions.

Given that the bosonic formula (1.18) significantly simplifies in the large k limit, it is of interest to look for an entirely analytic derivation of the equivalence of $\chi_{\ell,r}^{\mathrm{B}}(q)$ and $\chi_{\ell,r}^{\mathrm{F}}(q)$ in this limit. This will be the content of Section 4.

2. GRADED PARAFERMIONS AND $\widehat{\mathrm{osp}}(1,2)_k$ CHARACTERS

2.1. **Graded parafermions.** The graded parafermion conformal theory is defined by an algebra that generalizes the usual $\mathbb{Z}_k$ parafermionic algebra (i.e., $\psi_n \times \psi_m \sim \psi_{n+m}$ with $\psi_k \sim I$), by the addition of a $\mathbb{Z}_2$ grading $\psi_{1/2} \times \psi_{1/2} \sim \psi_1$ [7]. The new parafermion $\psi_{1/2}$ has conformal dimension $1 - 1/(4k)$. Associativity of the operator product algebra fixes the central charge as in (1.14) The parafermionic primary fields ϕ_ℓ are labeled by an integer ℓ such that $0 \le \ell \le k$. If we denote the

corresponding highest-weight state by $|\phi_\ell\rangle$, then modules over $|\phi_\ell\rangle$ can be decomposed into a finite sum of modules with specific charge. The latter is normalized by setting the charge of $\psi_{1/2}$ equal to one, and is in fact defined modulo $2k$ only, since $(\psi_{1/2})^{2k} \sim I$. The highest-weight state of relative charge r in the highest-weight module labeled by ℓ, has dimension

$$(2.1) \qquad h_\ell^{(r)} = \frac{\ell(2k - 3\ell)}{4k(2k + 3)} - \frac{r(2\ell + r)}{4k} + \max\{0, \lfloor (r + 1)/2 \rfloor\},$$

where $h_\ell^{(0)}$ will be simply denoted as h_ℓ.

In terms of the coset description (1.15), the label ℓ is the finite Dynkin label of $\widehat{\mathrm{osp}}(1, 2)_k$ and r is related to the u(1) charge m by

$$r = m - \ell.$$

With this identification, the dimension $h_\ell^{(r)}$ can be rewritten as

$$h_\ell^{(m-\ell)} = \frac{\ell(\ell + 1)}{2(2k + 3)} - \frac{m^2}{4k} + \max\{0, \lfloor (m - \ell + 1)/2 \rfloor\},$$

which can be recognized as the difference (modulo integers) between the dimension of the $\widehat{\mathrm{osp}}(1, 2)_k$ primary field with spin $\ell/2$ and that of the $\widehat{\mathrm{u}}(1)$ field of charge m.

In order to specify the highest-weight conditions, let us recall that when acting on a generic state of charge t, denoted by $|t\rangle$, the parafermionic modes B_n are defined as

$$\psi_{\frac{1}{2}}(z)|t\rangle = \sum_{m=-\infty}^{\infty} z^{-t/2k - m - 1} B_{(1+2t)/4k+m} |t\rangle,$$

$$\psi_{\frac{1}{2}}^\dagger(z)|t\rangle = \sum_{m=-\infty}^{\infty} z^{t/2k - m - 1} B_{(1-2t)/4k+m}^\dagger |t\rangle.$$

In the following, we use the more compact notation

$$\mathcal{B}_n|t\rangle := B_{n+(1+2t)/4k}|t\rangle, \qquad \mathcal{B}_n^\dagger|t\rangle := B_{n+(1-2t)/4k}^\dagger|t\rangle.$$

Then the defining relations for the highest-weight states $|\phi_\ell\rangle$ are

$$\mathcal{B}_n|\phi_\ell\rangle = 0 = \mathcal{B}_{n+1}^\dagger|\phi_\ell\rangle, \qquad n \geq 0.$$

The dimension h_ℓ of the highest-weight state $|\phi_\ell\rangle$ follows directly from these constraints together with the generalized commutation relations induced by the operator product algebra. The dimension $h_\ell^{(r)}$, in turn, is h_ℓ plus the dimension of $(\mathcal{B}_{-1})^r$ for $r > 0$ or $(\mathcal{B}_0^\dagger)^{-r}$ for $r < 0$.

2.2. **The Verma characters.** The free module over the highest-weight state $|\phi_\ell\rangle$ can be decomposed into a direct sum of modules with fixed relative charge r. The states of the module of charge $r \geq 0$ are of the form

$$(2.2) \qquad \mathcal{B}_{-\lambda_1}\mathcal{B}_{-\lambda_2}\cdots\mathcal{B}_{-\lambda_p}\mathcal{B}_{-\mu_1}^\dagger\mathcal{B}_{-\mu_2}^\dagger\cdots\mathcal{B}_{-\mu_{p'}}^\dagger|\phi_\ell\rangle,$$

where

$$(2.3) \qquad p - p' = r$$

and

(2.4a) $\qquad \lambda_i \geq \lambda_{i+1} - 1, \qquad\qquad \lambda_i \geq \lambda_{i+2}, \qquad\qquad \lambda_p \geq 1$

(2.4b) $\qquad \mu_i \geq \mu_{i+1} - 1, \qquad\qquad \mu_i \geq \mu_{i+2}, \qquad\qquad \mu_i \geq 0.$

Adopting the terminology of [20] we refer to the sequence $\lambda = (\lambda_1, \cdots, \lambda_p)$ as a jagged partition of length p and weight $\lambda_1 + \cdots + \lambda_p$. Clearly, the sequence $\mu = (\mu_1, \ldots, \mu_{p'})$ then corresponds to a jagged partition of length at most p'.

In order to compute the character $V_r(q)$ of the free module of relative charge r — Verma character for short — we denote by $J_{p,k}$ and $\bar{J}_{p',k} = \sum_{p=0}^{p'} J_{p,k}$ the number of jagged partitions of length p and the number of jagged partitions of length at most p'. For example, the four jagged partitions of length 4 and weight 5 are $(3,1,0,1)$, $(2,2,0,1)$, $(2,1,1,1)$ and $(1,2,1,1)$, so that $J_{4,5} = 4$. We further introduce the generating functions

$$J(z;q) = \sum_{p=0}^{\infty} J_p(q) z^p = \sum_{p,k=0}^{\infty} J_{p,k} z^p q^k$$

$$\bar{J}(z;q) = \sum_{p=0}^{\infty} \bar{J}_p(q) z^p = \sum_{p,k=0}^{\infty} \bar{J}_{p,k} z^p q^k$$

so that $J_p(q)$ and $\bar{J}_p(q)$ are the generating functions of jagged partitions of length p and of length at most p, respectively. Since $J_{p,k} = \bar{J}_{p,k} - \bar{J}_{p-1,k}$ we immediately infer that

(2.5) $$J(z;q) = (1-z)\bar{J}(z;q).$$

According to (2.3) and (2.4) the Verma character is expressed in terms of the above generating functions as

(2.6) $$V_r(q) = \sum_{\substack{p,p'=0 \\ p-p'=r}}^{\infty} J_p(q) \bar{J}_{p'}(q).$$

To obtain a closed form expression for $V_r(q)$ we need some standard q-series notation. Let $(a;q)_0 = 1$,

(2.7) $$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

for n a positive integer and

$$(a;q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \qquad |q| < 1.$$

Using that $(a;q)_n = (a;q)_{\infty}/(aq^n;q)_{\infty}$ one can extend the definition (2.7) to all integers n. In particular this implies that $1/(q;q)_{-n} = 0$ for n a positive integer.

Returning to our calculation of the Verma character we recall that [4]

(2.8) $$J(z;q) = \frac{(-zq;q)_{\infty}}{(z^2 q;q)_{\infty}}.$$

By Euler's two formulae [15, Eqs. (II.1) and (II.2)]

$$(2.9a) \qquad (z;q)_\infty = \sum_{j=0}^{\infty} \frac{(-z)^j q^{\binom{j}{2}}}{(q;q)_j},$$

$$(2.9b) \qquad \frac{1}{(z;q)_\infty} = \sum_{j=0}^{\infty} \frac{z^j}{(q;q)_j},$$

it thus follows that

$$(2.10) \qquad J_p(q) = \sum_{j=0}^{\infty} \frac{q^{\binom{p-2j+1}{2}+j}}{(q;q)_{p-2j}(q;q)_j}.$$

From (2.5) and (2.8) we find that

$$\bar{J}(z;q) = \frac{1}{1-z} \frac{(-zq;q)_\infty}{(z^2 q;q)_\infty} = \frac{(-z;q)_\infty}{(z^2;q)_\infty}.$$

Again using the Euler formulae this implies a companion to (2.10) as follows:

$$(2.11) \qquad \bar{J}_{p'}(q) = \sum_{j=0}^{\infty} \frac{q^{\binom{p'}{2}-2j}}{(q;q)_{p'-2j}(q;q)_j}.$$

An alternative derivation of (2.11) follows from Lemma 17 and Equations (56) and (58) of [13].

Substituting (2.10) and (2.11) in (2.6) results in

$$V_r(q) = \sum_{p,i,j=0}^{\infty} \frac{q^{\binom{p-2j+1}{2}+\binom{p-r-2i}{2}+j}}{(q;q)_{p-r-2i}(q;q)_i(q;q)_{p-2j}(q;q)_j}.$$

Shifting $p \to p + 2j$ and then summing over p using the Durfee rectangle identity [15, Eq. (II.8)]

$$(2.12) \qquad \sum_{k=0}^{\infty} \frac{q^{k(k+a)}}{(q;q)_k(q;q)_{k+a}} = \frac{1}{(q;q)_\infty}, \qquad a \in \mathbb{Z},$$

yields the expression for the Verma character of relative charge r given in (1.17). This should be compared with the quadruple sum representation for $V_r(q)$ of [4].

The Verma character is the character of a generic highest-weight module not containing singular vectors. Modules over $|\phi_\ell\rangle$ are highly reducible. A closed-form expression can be obtained for the singular vectors, from which their dimension and charge are easily read off. The irreducible modules are thus obtained by factoring these vectors using an exclusion-inclusion process. As a result, the characters of an irreducible module of relative charge r are expressed as the alternating sum of Verma characters given in (1.18).

2.3. The $\widehat{\mathrm{osp}}(1,2)_k/\hat{\mathrm{u}}(1)$ branching functions.
In this section we rederive (1.18) through a direct computation of the branching functions (1.15). This derivation is analogous to the one presented in the appendix for (1.10).

The character of the $\widehat{\mathrm{osp}}(1,2)_k$ integrable module indexed by the integer Dynkin label $0 \leq \ell \leq k$ is [10, App. A][1]

$$(2.13) \qquad \chi_\ell(x;q) = \frac{\Theta_{\ell+1/2}^{(k+3/2)}(x;q) - \Theta_{-\ell-1/2}^{(k+3/2)}(x;q)}{\Theta_{1/2}^{(3/2)}(x;q) - \Theta_{-1/2}^{(3/2)}(x;q)}.$$

Here $\Theta_m^{(k)}$ is the theta function

$$(2.14) \qquad \Theta_m^{(k)}(x;q) = \sum_{n=-\infty}^{\infty} q^{k\left(n+\frac{m}{2k}\right)^2} x^{-k\left(n+\frac{m}{2k}\right)}.$$

The $\widehat{\mathrm{u}}(1)$ character for a module with $\mathrm{u}(1)$ charge m is

$$K_m(x;q) = \frac{\Theta_m^{(k)}(x;q)}{\eta(q)}.$$

This form manifests the symmetry

$$K_m(x;q) = K_{m+2k}(x;q).$$

The claim that the graded parafermion characters arise as the branching functions of the $\widehat{\mathrm{osp}}(1,2)_k/\widehat{\mathrm{u}}(1)$ coset translates into the identity (1.16). The aim of this section is to show that this is in accordance with the bosonic representation of $\chi_{\ell,r}(q)$ as given in (1.18).

In passing we note that the $\widehat{\mathrm{osp}}(1,2)_k$ string functions are defined by the decomposition

$$\chi_\ell(x;q) = \sum_{m=1-k}^{k} c_m^\ell(q)\Theta_m^{(k)}(x;q),$$

so that (1.16) is equivalent to

$$\chi_{\ell,m-\ell}(q) = \eta(q)c_m^\ell(q).$$

Now, to obtain a formal expansion for the character $\chi_\ell(x;q)$ we use the quintuple product identity [15, Ex. 5.6]

$$\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}x^{3n}(1-xq^n) = (x, q/x, q; q)_\infty (qx^2, q/x^2; q^2)_\infty, \quad x \neq 0,$$

where $(a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n$, to rewrite the denominator of (2.13) (where we have set $x = y^2$) as

$$\Theta_{1/2}^{(3/2)}(y^2;q) - \Theta_{-1/2}^{(3/2)}(y^2;q) = q^{1/24}\, y^{-1/2} \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}y^{3n}(1-yq^n)$$

$$= q^{1/24}\, y^{-1/2}(y, q/y, q; q)_\infty(qy^2, q/y^2; q^2)_\infty$$

$$= q^{1/24}\, y^{-1/2}\frac{(y^2, q/y^2, q; q)_\infty}{(-y, -q/y; q)_\infty}.$$

By a double use of (2.9b) we obtain

$$(2.15) \qquad \frac{1}{(x, q/x; q)_\infty} = \sum_{i,j=0}^{\infty} \frac{q^j x^{i-j}}{(q;q)_i (q;q)_j}, \qquad |q| < |x| < 1.$$

---

[1]Recall that for osp(1,2), the dual Coxeter number is 3/2 and the Weyl vector is $\rho = \omega_1/2$ where ω_1 is the su(2) fundamental weight.

This and the Jacobi triple product identity [15, Eq. (II.28)]

$$(2.16) \qquad \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} x^n = (-x, -q/x, q; q)_\infty, \qquad x \neq 0,$$

imply that

$$\frac{(-y, -q/y; q)_\infty}{(y^2, q/y^2, q; q)_\infty} = \frac{1}{(q;q)_\infty^2} \sum_{n=-\infty}^{\infty} \sum_{i,j=0}^{\infty} \frac{y^{n+2i-2j} q^{\binom{n}{2}+j}}{(q;q)_i (q;q)_j}$$

$$= \frac{1}{(q;q)_\infty^2} \sum_{n=-\infty}^{\infty} \sum_{i,j=0}^{\infty} \frac{y^{-n} q^{\binom{2j-2i-n}{2}+j}}{(q;q)_i (q;q)_j}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} y^{-n} V_n(q), \qquad |q| < |y|^2 < 1,$$

where the last equality follows from (1.17). We thus arrive at the following expansion of the denominator of (2.13) in terms of Verma characters:

$$\frac{1}{\displaystyle\sum_{\sigma \in \{\pm 1\}} \Theta_{\sigma/2}^{(3/2)}(x;q)} = \frac{x^{1/4}}{\eta(q)} \sum_{n=-\infty}^{\infty} x^{-n/2} V_n(q).$$

Multiplying the above expression by the numerator of (2.13) yields

$$\chi_\ell(x;q) = \frac{1}{\eta(q)} \sum_{\sigma \in \{\pm 1\}} \sigma \sum_{j,n=-\infty}^{\infty} x^{-n/2 + j(2k+3)/2 + \sigma(\ell+1/2)/2 + 1/4}$$

$$\times q^{\frac{1}{2}(2k+3)\left(\sigma j + \frac{2\ell+1}{2(2k+3)}\right)^2} V_n(q).$$

We now arrange the x dependence into sums over the $\widehat{\mathrm{u}}(1)$ characters. Shifting $n \to n - (2k+3)j - \sigma(\ell+1/2) + 1/2$ and replacing j by σj this is achieved as

$$\chi_\ell(x;q) = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} x^{-n/2} \sum_{j=-\infty}^{\infty} q^{\frac{1}{2}(2k+3)\left(j + \frac{2\ell+1}{2(2k+3)}\right)^2}$$

$$\times \sum_{\sigma \in \{\pm 1\}} \sigma V_{n - \sigma(2k+3)j - \sigma(\ell+1/2)+1/2}(q)$$

$$= \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} x^{-n/2} q^{n^2/4k} \chi_{\ell, n-\ell}(q).$$

with $\chi_{\ell,r}(q)$ given by (1.18). To complete the decomposition of the $\widehat{\mathrm{osp}}(1,2)_k$ character, we set $n = 2kj + m$ with $1 - k \leq m \leq k$ and use the symmetry $\chi_{\ell,m}(q) = \chi_{\ell,m+2kj}(q)$. This gives

$$\chi_\ell(x;q) = \sum_{m=1-k}^{k} \chi_{\ell,m-\ell}(q) \sum_{j=-\infty}^{\infty} \frac{q^{k(j+m/2k)^2} x^{-k(j+m/2k)}}{\eta(q)}$$

$$= \sum_{m=1-k}^{k} \chi_{\ell,m-\ell}(q) K_m(x;q)$$

in agreement with (1.18) and (1.16).

2.4. **Fermionic parafermion characters.** Fermionic forms for the characters of the graded parafermion models were first constructed in [4]. Correcting a minor misprint, the result of [4] states that

$$(2.17) \quad \chi^{\mathrm{F}}_{\ell,r}(q) = q^{h_\ell - c/24}$$

$$\times \sum_{\substack{n_0,\dots,n_{k-1}=0 \\ n_0+2N\equiv r \ (\mathrm{mod}\ 2k)}}^{\infty} \frac{q^{\binom{n_0+1}{2}+N_1^2+\cdots+N_{k-1}^2+N_{k-\ell+1}+\cdots+N_{k-1}-(n_0+2N)(n_0+2N+2\ell)/(4k)}}{(q;q)_{n_0}\cdots(q;q)_{n_{k-1}}},$$

with $N_i = n_i + \cdots + n_{k-1}$, $N = N_1 + \cdots + N_{k-1}$ and $N_k = N_{k+1} = 0$.

The above fermionic form encodes the combinatorics of the graded parafermionic quasi-particle basis constructed in [20]. This basis is described by states of the form

$$\mathcal{B}_{-\lambda_1}\mathcal{B}_{-\lambda_2}\cdots\mathcal{B}_{-\lambda_p}|\phi_\ell\rangle$$

for $p = mk + r$. Here the λ_i are again subject to (2.4) with $\lambda_p \geq 1$, but — in the case of strictly positive k — are further constrained by

$$\lambda_j \geq \lambda_{j+2k-1} + 1 \qquad \text{or} \qquad \lambda_j = \lambda_{j+1} - 1 = \lambda_{j+2k-2} + 1 = \lambda_{j+2k-1},$$

for all $j \leq p - 2k + 1$.

In the derivation of (2.17) it is understood that $r \geq 0$ (that is, we act with r modes modulo k on the highest-weight state). We shall later show, however, that the expression (2.17) obeys the formal symmetry relation

$$(2.18) \qquad\qquad \chi_{\ell,r}(q) = \chi_{\ell,-r-2\ell}(q).$$

This will allow us to interpret $\chi_{\ell,r}(q)$ for all integers r.

To connect with the fermionic form for the characters stated in (1.19) we first need to replace $n_i \to n_{k-i}$ for $1 \leq i \leq k-1$ and use the symmetry (2.18). Then introducing the vector notation

$$wAv = \sum_{i,j=1}^{k-1} w_i A_{ij} v_j \qquad \text{and} \qquad (Av)_i = \sum_{j=1}^{k-1} A_{ij} v_j$$

for $w, v \in \mathbb{Z}^{k-1}$ and A a square matrix of dimension $k-1$, we get (1.19) with C the A_{k-1} Cartan matrix:

$$(2.19) \qquad\qquad C_{ij}^{-1} = \min\{i,j\} - \frac{ij}{k},$$

e_i the ith standard unit vector in $\mathbb{Z}^{k-1}$ (with $e_0 = e_k$ the zero vector) and $n = (n_1,\dots,n_{k-1})$.

We note that since the entries of C^{-1} are integer multiples of $1/k$, the restriction imposed on the sum over the n_i in (1.19) implies that the summand vanishes unless $n_0 - r$ is even.

3. Proof of B = S = F for graded parafermions

This section, in which we prove the equivalence of the bosonic, fermionic and spinon forms of the graded parafermion characters, forms the core of the paper.

3.1. **Main results.** Normalising the characters in (1.13) we set out to prove the following theorem. Let

$$(3.1) \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & k \in \{0, 1, \dots, n\}, \\[2mm] 0 & \text{otherwise} \end{cases}$$

be a q-binomial coefficient, and assume the vector notation introduced in Section 2.4.

Theorem 3.1. *Let k be a positive integer. For $\ell \in \{0, \dots, k\}$ and $r \in \mathbb{Z}$ there holds*

$$(3.2) \quad \sum_{j=-\infty}^{\infty} q^{j((2k+3)j+2\ell+1)/2} \Big\{ V_{r-\ell-(2k+3)j}(q) - V_{r+\ell+1+(2k+3)j}(q) \Big\}$$

$$= (q;q)_\infty \sum_{\substack{L,n_0,\dots,n_{k-1}=0 \\ \frac{2L+r+\ell+n_0}{2k}+(C^{-1}n)_1 \in \mathbb{Z}}}^{\infty} q^{L(L+r)/k+(r-\ell-n_0)(r+\ell+n_0)/(4k)+\binom{n_0+1}{2}}$$

$$\times \frac{q^{nC^{-1}(n-n_0 e_1 - e_\ell)}}{(q;q)_L (q;q)_{L+r}(q;q)_{n_0}} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}$$

$$= \sum_{\substack{n_0,\dots,n_{k-1}=0 \\ \frac{r+\ell+n_0}{2k}+(C^{-1}n)_1 \in \mathbb{Z}}}^{\infty} \frac{q^{(r-\ell-n_0)(r+\ell+n_0)/(4k)+\binom{n_0+1}{2}+nC^{-1}(n-n_0 e_1 - e_\ell)}}{(q;q)_{n_0} \cdots (q;q)_{n_{k-1}}}.$$

Here the m_i in the expression after the first equality follow from

$$(3.3) \qquad m = C^{-1}\Big[(2L+r+n_0)e_1 + e_\ell - 2n \Big].$$

Similar to our earlier comment we remark that the summands of the two multiple sums on the right vanish when $n_0 + r + \ell \not\equiv 0 \pmod 2$.

For the sake of brevity we write the statement of Theorem 3.1 as

$$(3.4) \qquad \psi_{\ell,r}^{\mathrm{B}}(q) = \psi_{\ell,r}^{\mathrm{S}}(q) = \psi_{\ell,r}^{\mathrm{F}}(q).$$

Comparison with (1.18), (1.19) and (1.21) shows that

$$(3.5) \qquad \chi_{\ell,r-\ell}(q) = q^{h_\ell - (r-\ell)(r+\ell)/(4k) - c/24} \psi_{\ell,r}(q)$$

so that (3.4) implies (1.13).

The remainder of this section will be devoted to a proof of Theorem 3.1, partially by analytic and partially by combinatorial means. Because of the length and complexity of the proof, we will state the main intermediate results leading to the theorem in the form of three propositions. Section 3.2 will provide some further insight into the origin of the first proposition (which essentially is a restatement of (1.27)). Then, in Sections 3.3–3.5, each of the propositions will be proved.

Before we can state our first proposition we need to recall the definition of the one-dimensional configuration sums of the Andrews–Baxter–Forrester model, given

by [2, 12]

$$(3.6) \quad X_{r,s}^{(p,p')}(L,b;q) = \sum_{j=-\infty}^{\infty} \left\{ q^{j(pp'j+p'r-ps)} \begin{bmatrix} L \\ (L+s-b)/2 - p'j \end{bmatrix} \right.$$
$$\left. - q^{(pj+r)(p'j+s)} \begin{bmatrix} L \\ (L-s-b)/2 - p'j \end{bmatrix} \right\}.$$

Here p, p', r, s, b and L are integers such that $1 \le p \le p'$, $1 \le b, s \le p'-1$, $0 \le r \le p$, and $L + s + b \equiv 0 \pmod{2}$.

Our first proposition allows for the rewriting of the left-hand side of (3.4) in terms of one-dimensional configuration sums, and may be recognized as a normalized version of (1.27).

Proposition 3.1. *For r an integer and $\ell \in \{0, \ldots, k\}$ there holds*

$$\psi_{\ell,r}^{\mathrm{B}}(q) = (q;q)_\infty \sum_{\substack{n=0 \\ n \equiv r+\ell \pmod{2}}}^{\infty} \frac{q^{n^2/2}}{(q;q)_n} \sum_{L=0}^{\infty} \frac{q^{(2L+r-\ell)/2}}{(q;q)_L (q;q)_{L+r}}$$
$$\times \sum_{\substack{i=0 \\ i \equiv r \pmod{2}}}^{k} X_{1,i+1}^{(1,k+2)}(n, \ell+1; q^{-1}) X_{1,i+1}^{(1,k+2)}(2L+r, 1; q).$$

Since $1/(q;q)_{-n} = 0$ for n a positive integer, the lower bound in the sum over L may be replaced by $\max\{0, -r\}$. By shifting the summation index $L \to L - r$ it thus follows that $\psi_{\ell,r}(q) = \psi_{\ell,-r}(q)$. By (3.5) this implies $\chi_{\ell,r-\ell}(q) = \chi_{\ell,-r-\ell}(q)$ thereby establishing the previously claimed symmetry (2.18).

The next proposition provides a fermionic representation for the second line in the above expression.

Proposition 3.2. *For L, M integers and $\ell \in \{0, \ldots, k\}$ such that*

$$(3.7a) \qquad\qquad\qquad L + M + \ell \equiv 0 \pmod{2}$$

there holds

$$(3.7b) \qquad \sum_{\substack{i=0 \\ i \equiv L \pmod{2}}}^{k} X_{1,i+1}^{(1,k+2)}(M, \ell+1; q^{-1}) X_{1,i+1}^{(1,k+2)}(L, 1; q)$$

$$= q^{(L^2 - (M+\ell)^2)/(4k) - (L-M-\ell)/2} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{L+M+\ell}{2k} + (C^{-1}n)_1 \in \mathbb{Z}}} q^{nC^{-1}(n - Me_1 - e_\ell)} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}.$$

Here the m_i follow from

$$(3.7c) \qquad\qquad\qquad m = C^{-1} \Big[(L+M)e_1 + e_\ell - 2n \Big].$$

Clearly, combining Proposition 3.1 (with $n \to n_0$) and Proposition 3.2 (with $M \to n_0$ and $L \to 2L + r$) results in

$$\psi_{\ell,r}^{\mathrm{B}}(q) = \psi_{\ell,r}^{\mathrm{S}}(q).$$

Our third and final proposition is essentially a formula from [28].

Proposition 3.3. *For r, n_0 integers and $\ell = \{0, \ldots, k\}$ such that*

$$n_0 + r + \ell \equiv 0 \pmod 2$$

there holds

$$\sum_{L=0}^{\infty} \frac{q^{L(L+r)/k}}{(q;q)_L (q;q)_{L+r}} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{2L+r+\ell+n_0}{2k} + (C^{-1}n)_1 \in \mathbb{Z}}} q^{nC^{-1}(n - n_0 e_1 - e_\ell)} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}$$

$$= \frac{1}{(q;q)_\infty} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{r+\ell+n_0}{2k} + (C^{-1}n)_1 \in \mathbb{Z}}} \frac{q^{nC^{-1}(n - n_0 e_1 - e_\ell)}}{(q;q)_{n_1} \cdots (q;q)_{n_{k-1}}},$$

with the m_i determined by (3.3).

Applying the above to the first expression on the right of (3.2), i.e., to $\psi_{\ell,r}^{\mathrm{S}}(q)$, yields

$$\psi_{\ell,r}^{\mathrm{S}}(q) = \psi_{\ell,r}^{\mathrm{F}}(q).$$

3.2. **From** (1.26) **to** (1.27). In this preliminary section, which is not part of the proof of Theorem 3.1, we complete our earlier discussion and show how the coset decomposition (1.22) naturally gives rise to the character expression (1.27). This in particular motivates Proposition 3.1 and, to a lesser extend, Proposition 3.2.

We already sketched (no actual proof of (1.25) has been given) how (1.22) leads to (1.26). To transform this into (1.27) we need to express both characters in the summand on the right in terms of one-dimensional configuration sums.

In the case of the Virasoro characters $\tilde{\chi}_{r,s}^{(p,p')}(q)$ this is not difficult. Of course the simplest connection between Virasoro characters and configuration sums follows by taking the large L limit in the latter:

$$\chi_{r,s}^{(p,p')}(q) = \lim_{L \to \infty} X_{r,s}^{(p,p')}(L, b; q) = \sum_{j=-\infty}^{\infty} \left\{ q^{j(pp'j + p'r - ps)} - q^{(pj+r)(p'j+s)} \right\}.$$

Here $\chi_{r,s}^{(p,p')}(q) = 1 + O(q)$ is the normalised character

(3.8) $\tilde{\chi}_{r,s}^{(p,p')}(q) = q^{h_{r,s}^{(p,p')} - c^{(p,p')}/24} \chi_{r,s}^{(p,p')}(q),$

with

$$c^{(p,p')} = 1 - \frac{6(p - p')^2}{pp'}$$

$$h_{r,s}^{(p,p)} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}$$

the central charge and conformal weights of the minimal model $\mathcal{M}(p, p')$.

There is however another way in which the Virasoro characters follow from the configuration sums, and by a straightforward application of (2.12) and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

one finds

$$(3.9) \qquad \chi_{s,2b-r}^{(p',2p'-p)}(q) = q^{-(s-b)^2/2} \sum_{\substack{n=0 \\ n \equiv s+b \ (\mathrm{mod}\ 2)}}^{\infty} \frac{q^{n^2/2}}{(q;q)_n} X_{r,s}^{(p,p')}(n,b;q^{-1}).$$

The problem of expressing the parafermion character $\hat{\chi}_{\ell,r}(q)$ in terms of configuration sums is a lot more difficult, but has been fully resolved in [29, Cor. 4.2]. Using this result with $p = 1$ and $p' = k+2$, we find

$$\hat{b}_{i,r}(q) = (q;q)_\infty q^{\Delta_i + (i^2 - r^2)/(4k)} \sum_{L=0}^{\infty} \frac{X_{0,i+1}^{(p,p')}(2L+r,1;q)}{(q;q)_L (q;q)_{L+r}}$$

with Δ_i given by (1.7). Also using [29, Eq. (2.9)]

$$(3.10) \qquad X_{0,s}^{(p,p')}(L,1;q) = q^{(L-s+1)/2} X_{1,s}^{(p,p')}(L,1;q)$$

and (1.3) this yields

$$\hat{\chi}_{i,\frac{r-i}{2}}(q) = (q;q)_\infty q^{\Delta_i + (i^2 - r^2)/(4k)} \sum_{L=0}^{\infty} \frac{q^{(2L+r-i)/2} X_{1,i+1}^{(1,k+2)}(2L+r,1;q)}{(q;q)_L (q;q)_{L+r}}.$$

Substituting this as well as (3.9) (with $p = 1$, $p' = k+2$, $r = 1$, $s = i+1$ and $b = \ell + 1$) into (1.26) (to this end we also need (3.8)) results in (1.27) since

$$h_{i+1,2\ell+1}^{(k+2,2k+3)} - \frac{1}{24}c^{(k+2,2k+3)} + \Delta_i + \frac{i^2}{4k} - \binom{\ell - i}{2} = h_\ell + \frac{\ell^2}{4k} - \frac{c}{24}.$$

3.3. **Proof of Proposition 3.1.** By (3.9) and (3.10) Proposition 3.1 can be simplified to

$$\psi_{\ell,r}^{\mathrm{B}}(q) = (q;q)_\infty \sum_{L=0}^{\infty} \frac{1}{(q;q)_L (q;q)_{L+r}}$$

$$\times \sum_{\substack{i=0 \\ i \equiv r \ (\mathrm{mod}\ 2)}}^{k} q^{\binom{\ell-i}{2}} \chi_{i+1,2\ell+1}^{(k+2,2k+3)}(q) X_{0,i+1}^{(1,k+2)}(2L+r,1;q).$$

Next we apply the following lemma with $p = 1$ and $p' = k+2$.

Lemma 3.1. *For $P = p'$ and $P' = 2p' - p$,*

$$(3.11) \qquad \sum_{\substack{i=0 \\ i \equiv r \ (\mathrm{mod}\ 2)}}^{p'-2} q^{\binom{\ell-i}{2}} \chi_{i+1,2\ell+1}^{(P,P')}(q) X_{0,i+1}^{(p,p')}(2L+r,1;q)$$

$$= \frac{1}{(q;q)_\infty} \sum_{j,n=-\infty}^{\infty} q^{pj(P'j+2\ell+1)/2+\binom{2n+\ell-r+P'j}{2}} \left\{ \begin{bmatrix} 2L+r \\ L+n \end{bmatrix} - \begin{bmatrix} 2L+r \\ L+n-1 \end{bmatrix} \right\}.$$

Hence,

$$\psi_{\ell,r}^{\mathrm{B}}(q) = \sum_{j,n=-\infty}^{\infty} q^{j((2k+3)j+2\ell+1)/2+\binom{2n+\ell-r+(2k+3)j}{2}}$$

$$\times \sum_{L=0}^{\infty} \frac{1}{(q;q)_L (q;q)_{L+r}} \left\{ \begin{bmatrix} 2L+r \\ L+n \end{bmatrix} - \begin{bmatrix} 2L+r \\ L+n-1 \end{bmatrix} \right\}.$$

This can be further transformed by yet another lemma.

Lemma 3.2. *For n and r integers,*

$$\sum_{L=0}^{\infty} \frac{1}{(q;q)_L (q;q)_{L+r}} \left\{ \begin{bmatrix} 2L+r \\ L+n \end{bmatrix} - \begin{bmatrix} 2L+r \\ L+n-1 \end{bmatrix} \right\}$$

$$= \frac{1}{(q;q)_\infty} \sum_{j=0}^{\infty} \left\{ \frac{q^j}{(q;q)_j (q;q)_{j-n}} - \frac{q^j}{(q;q)_j (q;q)_{j+n-r-1}} \right\}.$$

Thanks to this second lemma we are left with

$$\psi_{\ell,r}^{\mathrm{B}}(q) = \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} q^{j((2k+3)j+2\ell+1)/2}$$

$$\times \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} q^{\binom{2n+\ell-r+(2k+3)j}{2}} \left\{ \frac{q^m}{(q;q)_m (q;q)_{m-n}} - \frac{q^m}{(q;q)_m (q;q)_{m+n-r-1}} \right\}.$$

Finally replacing $n \to m-n$ and $n \to n-m+r+1$ respectively in the two sums over n on the right, and recalling (1.17), we find

$$\psi_{\ell,r}^{\mathrm{B}}(q) = \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} q^{j((2k+3)j+2\ell+1)/2} \left\{ V_{r-\ell-(2k+3)j}(q) - V_{r+\ell+1+(2k+3)j}(q) \right\}$$

in accordance with the left-hand side of (3.2).

Of course we still need to prove Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. The proof of (3.11) is rather simple. In the double sum on the right we replace $n \to p'n + (r-i)/2$ followed by $j \to j-n$. Hence, since $p' = P$,

$$\mathrm{RHS}(3.11) = \frac{1}{(q;q)_\infty} \sum_{\substack{i=0 \\ i \equiv r \pmod 2}}^{2p'-2} q^{\binom{\ell-i}{2}} \sum_{j=-\infty}^{\infty} q^{j(PP'j+P(2\ell+1)-P'(i+1))}$$

$$\times \sum_{n=-\infty}^{\infty} \left\{ q^{pn(p'n-i-1)} \left\{ \begin{bmatrix} 2L+r \\ (2L+r-i)/2+p'n \end{bmatrix} \right.\right.$$

$$\left.\left. - \begin{bmatrix} 2L+r \\ (2L+r-i-2)/2+p'n \end{bmatrix} \right\} \right\}$$

$$= \frac{1}{(q;q)_\infty} \sum_{\substack{i=0 \\ i \equiv r \pmod 2}}^{2p'-2} q^{\binom{\ell-i}{2}} X_{0,i+1}^{(p,p')}(2L+r,1;q)$$

$$\times \sum_{j=-\infty}^{\infty} q^{j(PP'j+P(2\ell+1)-P'(i+1))}.$$

Next we use

$$\sum_{\substack{i=0 \\ i \equiv r \pmod 2}}^{2p'-2} f_{i+1} = \sum_{\substack{i=0 \\ i \equiv r \pmod 2}}^{p'-2} (f_{i+1} + f_{2p'-i-1})$$

provided that $f_{p'} = 0$. Since

(3.12) $$X_{0,p'}^{(p,p')}(L,1;q) = 0$$

this may be applied to the sum over i. Also using

$$X_{0,i+1}^{(p,p')}(L,1;q) = -q^{p(p'-i-1)}X_{0,2p'-i-1}^{(p,p')}(L,1;q)$$

(this in fact implies (3.12)) yields

$$\text{RHS}(3.11) = \frac{1}{(q;q)_\infty} \sum_{\substack{i=0 \\ i \equiv r \pmod 2}}^{p'-2} q^{\binom{\ell-i}{2}} X_{0,i+1}^{(p,p')}(2L+r,1;q)$$

$$\times \sum_{j=-\infty}^{\infty} \left\{ q^{j(PP'j+P(2\ell+1)-P'(i+1))} - q^{(P(j-1)+i+1)(P'(j-1)+2\ell+1)} \right\}.$$

Replacing $j \to j+1$ in the second term in the sum over j completes the proof. $\square$

Proof of Lemma 3.2. We will show that
(3.13)

$$(a-b)\sum_{j=0}^{\infty} \frac{(ab;q)_{2j}\, q^j}{(q,aq,bq,ab;q)_j} = \sum_{j=0}^{\infty} q^j \left\{ \frac{a}{(q,aq;q)_j(bq;q)_\infty} - \frac{b}{(q,bq;q)_j(aq;q)_\infty} \right\}.$$

Letting $a \to aq^n$, $b \to bq^{r-n+1}$ this gives

$$\sum_{j=0}^{\infty} \frac{(abq;q)_{2j+r}(aq^{j+n} - bq^{j+r-n+1})}{(q;q)_j(abq;q)_{j+r}(aq;q)_{j+n}(bq;q)_{j+r-n+1}}$$

$$= \sum_{j=0}^{\infty} \left\{ \frac{aq^{j+n}}{(q;q)_j(aq;q)_{j+n}(bq;q)_\infty} - \frac{bq^{j+r-n+1}}{(q;q)_j(bq;q)_{j+r-n+1}(aq;q)_\infty} \right\}.$$

Shifting $j \to j-n$ and $j \to j-r+n-1$ in the two terms on the right and putting $a=b=1$ yields the claim of the lemma since

$$\frac{(q^{j+n} - q^{j+r-n+1})(q;q)_{2j+r}}{(q;q)_{j+n}(q;q)_{j+r-n+1}} = \begin{bmatrix} 2j+r \\ j+n \end{bmatrix} - \begin{bmatrix} 2j+r \\ j+n-1 \end{bmatrix}.$$

Equation (3.13) is similar to [29, Eq. (4.3)] and [32, Thm 1.5] and the proof proceeds accordingly. First we multiply both sides of (3.13) by $(aq,bq;q)_\infty$ and use $(a;q)_\infty/(a;q)_n = (aq^n;q)_\infty$ to obtain

$$(a-b)\sum_{j=0}^{\infty} \frac{q^j(abq^j;q)_j(aq^{j+1},bq^{j+1};q)_\infty}{(q;q)_j} = \sum_{j=0}^{\infty} q^j \left\{ \frac{a(aq^{j+1};q)_\infty}{(q;q)_j} - \frac{b(bq^{j+1};q)_\infty}{(q;q)_j} \right\}.$$

Next we expand each of the q-shifted factorials depending on a and/or b by (2.9a) or the q-binomial theorem [1, Eq. (3.3.6)]

$$\sum_{n=0}^{\infty} (-1)^n a^n q^{\binom{n}{2}} \begin{bmatrix} M \\ n \end{bmatrix} = (a;q)_M.$$

This results in

$$\sum_{j,k,l,n=0}^{\infty} \frac{(-1)^{k+l+n} a^{k+n} b^{l+n} q^{\binom{k+1}{2}+\binom{l+1}{2}+\binom{n}{2}+j(k+l+n+1)}}{(q;q)_n(q;q)_{j-n}(q;q)_k(q;q)_l}$$

$$= \sum_{j,k=0}^{\infty} \frac{a^{k+1} - b^{k+1}}{a-b} \frac{(-1)^k q^{\binom{k+1}{2}+j(k+1)}}{(q;q)_j(q;q)_k}.$$

After the shift $j \to j+n$ on the left, both sums over j can be carried out by (2.9b), leading to

$$\sum_{k,l,n=0}^{\infty} \frac{(-1)^{k+l+n} a^{k+n} b^{l+n} q^{\binom{k+1}{2}+\binom{l+1}{2}+\binom{n}{2}+n(k+l+n+1)} (q;q)_{k+l+n}}{(q;q)_n (q;q)_k (q;q)_l}$$

$$= \sum_{k=0}^{\infty} \frac{a^{k+1}-b^{k+1}}{a-b} (-1)^k q^{\binom{k+1}{2}}.$$

Equating coefficients of $a^k b^l$ and performing some standard manipulations yields

$$_2\phi_1(q^{-k},q^{-l};q^{-k-l};q,1) = q^{kl} \frac{(q;q)_k (q;q)_l}{(q;q)_{k+l}}.$$

Here

$$_{r+1}\phi_r \begin{bmatrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{bmatrix} ; q, z \end{bmatrix}$$

$$= {}_{r+1}\phi_r(a_1,\cdots,a_{r+1};b_1,\cdots,b_r;q,z) = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_{r+1};q)_n}{(b_1\cdots,b_r,q;q)_n} z^n$$

is a basic hypergeometric series, see [15]. Summing the $_2\phi_1$ series using the q-Chu–Vandermonde sum [15, Eq. (II.7)]

$$_2\phi_1(a,q^{-n};c;q,cq^n/a) = \frac{(c/a;q)_n}{(c;q)_n}$$

completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is interesting to note that (3.13) admits the following bounded analogue

$$(a-b) \sum_{j=0}^{M} q^j \begin{bmatrix} M \\ j \end{bmatrix} \frac{(ab;q)_{2j}}{(aq,bq,ab;q)_j}$$

$$= \sum_{j=0}^{M} q^j \begin{bmatrix} M \\ j \end{bmatrix} \left\{ \frac{a}{(aq;q)_j (bq;q)_{M-j}} - \frac{b}{(bq;q)_j (aq;q)_{M-j}} \right\}.$$

We leave its proof (which is surprisingly difficult) to the reader.

3.4. **Proof of Proposition 3.2.** In [30, 31] a combinatorial technique was developed to obtain fermionic representations for the one-dimensional configuration sums

$$X_{r,s}^{(p,p+1)}(L,b;q).$$

Combining this technique with the work of Berkovich and Paule [5] on generalizations of the Andrews–Gordon identities results in Proposition 3.2. More precisely, we will first show — following the method of [5] — that (after a simple rewriting) the left-hand side of (3.7b) may be interpreted as the generating function of a particular set of lattice paths. We will then use the method of [30, 31] to show that this generating function permits a fermionic form in accordance with the right-hand side of (3.7b). The details of the proof marginally differ according to whether $\ell = 0$ or $\ell \in \{1,\ldots,k\}$. We will first treat the $\ell = 0$ case in full detail and then point out the relevant differences with $\ell \in \{1,\ldots,k\}$.

Our initial step is to take (3.7b) with $\ell = 0$ and use [29, Eq. (2.3)]

$$(3.14) \qquad X_{r,s}^{(p,p')}(L,b;q) = q^{(L^2-(b-s)^2)/4} X_{b-r,s}^{(p'-p,p')}(L,b;q^{-1})$$

followed by (3.10) to transform the left-hand side. On the right-hand side we use (3.7c) to eliminate n in the exponent of q in favour of m. As a result we obtain the $\ell = 0$ instance of the more general identity

$$(3.15a) \qquad \sum_{\substack{i=0 \\ i\equiv L \pmod 2}}^{k} q^{\ell(\ell-2i+1)/4} X_{\ell+1,i+1}^{(k+1,k+2)}(M,\ell+1;q) X_{1,i+1}^{(k+1,k+2)}(L,1;q^{-1})$$

$$= \sum_{\substack{n\in\mathbb{Z}^{k-1} \\ \frac{L+M-\ell}{2k}+(C^{-1}n)_1\in\mathbb{Z}}} q^{\frac{1}{4}mCm-\frac{1}{2}Lm_1} \prod_{i=1}^{k-1} \begin{bmatrix} n_i+m_i \\ n_i \end{bmatrix},$$

where

$$(3.15b) \qquad m = C^{-1}\Big[(L+M)e_1 + e_{k-\ell} - 2n\Big]$$

and

$$(3.15c) \qquad L+M+\ell \equiv 0 \pmod 2$$

and

$$(3.15d) \qquad \ell \in \{0,\ldots,k-1\}.$$

What we will set out to do is to compute the generating function of lattice paths contained in a strip of height k starting in $(-L,0)$ and terminating with a step from (M,ℓ) to $(M+1,\ell+1)$. More precisely, starting from $(-L,0)$ we carry out a sequence of north-east (up or $+$) and south-east (down or $-$) steps (an up step is from (x,y) to $(x+1,y+1)$ and a down step is from (x,y) to $(x+1,y-1)$, with x,y integers) such that the y coordinate of each vertex along the path is always nonnegative and never exceeds k, and such that the last step is in the north-east direction and terminates in $(M+1,\ell+1)$. An example of such a lattice path is shown in Figure 1. Obviously there are no admissible paths unless (3.15c) and (3.15d) are satisfied. The *shape* of a path is given by its sequence of consecutive steps, ignoring the actual positions of the path's starting and ending vertices. The shape of the path of Figure 1 is

$$(+,-,+,+,-,+,+,+,-,-,-,-,+,+,+,+,-,-,+,+).$$

To a path P we assign a weight (or statistic) $W(P)$ given by the sum of the x-coordinates of vertices of the type shown in Figure 2. For example, the weight P of the path of Figure 1 is given by

$$0+3+4+6+7+8+10+11+12+14+16.$$

The aim is to compute the generating function

$$G_\ell(L,M;q) = \sum_P q^{W(P)/2},$$

where the sum is over all admissible paths P. The rationale behind the half in the exponent on the right is the fact that $W(P) \equiv \binom{\ell+1}{2} - \ell L \pmod 2$ so that, up to a possible overall factor $q^{1/2}$, $G_\ell(L,M;q)$ is a Laurent polynomial in q.
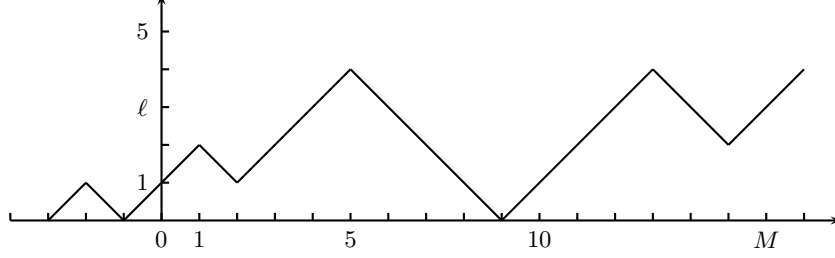
FIGURE 1. Example of path P with $k \geq 4$, $L = 3$, $M = 16$ and $\ell = 3$.

FIGURE 2. Vertices of weight x.

Our first method of computing $G_\ell(L, M; q)$ closely follows [5], and, as we shall see shortly, gives the left-hand side of (3.15a). First let us introduce the analogous problem of computing lattice paths in the strip with the same bounds on the y coordinates but with initial vertex $(0, i)$ and final two vertices (L, ℓ), $(L + 1, \ell + 1)$. Defining the same statistic as before we denote the generating function of such paths by $G_{i,\ell}(L; q)$.

After L steps a path P from $(-L, 0)$ to (M, ℓ), $(M + 1, \ell + 1)$ crosses the non-negative y-axis at some vertex $(0, i)$, where $0 \leq i \leq k$ and $i \equiv L \pmod 2$. Because this vertex has x-coordinate equal to zero, the 'local shape' (i.e., $(+, +)$, $(+, -)$, $(-, +)$ or $(-, -)$) of the paths when crossing the y-axis will not affect the weight $W(P)$. Consequently, one may view P as the concatenation of two lattice paths P_+ and P_-, the former starting at $(0, i)$, pointing in the positive x-direction and terminating in (M, ℓ), $(M + 1, \ell + 1)$ and the latter starting at $(0, i)$, pointing in the negative x-direction, and terminating in $(-L, 0)$, $(-L - 1, 1)$. (The addition of the final up-step from $(-L, 0)$ to $(-L - 1, 1)$ is permitted since it will not change the weight of P_-. Indeed the final two steps of P_- will now be $(-, +)$). The path P_- may be reflected in the y-axis so that it becomes a path pointing in the positive x-direction, starting at $(0, i)$ and terminating in $(L, 0)$ to $(L + 1, 1)$. Since the reflection negates the weight of P_- we need to replace q by q^{-1} in the corresponding generating function of P_- type paths. At the level of the generating function the

FIGURE 3. Example of path P with particle content $(n_1, n_2, n_3, n_4, \dots) = (2, 5, 0, 1, 0, 0, \dots)$.

above decomposition thus becomes

$$G_\ell(L, M; q) = \sum_{\substack{i=0 \\ i \equiv L \pmod{2}}}^{k} G_{i,\ell}(M; q) G_{i,0}(L; q^{-1}).$$

Since $G_{i,\ell}(0; q) = \delta_{i,\ell}$ this includes the obvious relations

$$G_\ell(0, M; q) = G_{0,\ell}(M; q) \quad \text{and} \quad G_\ell(L, 0; q) = G_{\ell,0}(L; q^{-1}).$$

The $G_{i,\ell}(L; q)$ may be computed from a simple recursion, and quoting the result of [2] we have

(3.16) $$G_{i,\ell}(L; q) = q^{(i-\ell)(i-\ell-1)/4} X_{\ell+1,i+1}^{(k+1,k+2)}(L, \ell+1; q).$$

The generating function $G_\ell(L, M; q)$ is thus found to be

$$G_\ell(L, M; q) = \sum_{\substack{i=0 \\ i \equiv L \pmod{2}}}^{k} q^{\ell(\ell-2i+1)/4} X_{\ell+1,i+1}^{(k+1,k+2)}(M, \ell+1; q) X_{1,i+1}^{(k+1,k+2)}(L, 1; q^{-1})$$

in accordance with the left-hand side of (3.15a).

The second method of computing $G_\ell(L, M; q)$ closely follows [30, 31] where it was used to find fermionic representations for the generating function $G_{i,\ell}(L; q)$. The method amounts to interpreting an admissible path as a collection of charged particles with charges taken from the set $\{1, \dots, k\}$ (the charge being the effective height of a particle). Hence each admissible path P may be assigned a sequence of integers $(n_1, \dots, n_k)$ where n_j is the number of particles of charge j in P. For example, the decomposition into particles of the path in Figure 3 is indicated by the dotted lines, and this particular path has $n_1 = 2$, $n_2 = 5$ and $n_4 = 1$ and all other $n_j = 0$. Because a path has fixed initial and final vertices, the n_j are subject to the constraint

$$2 \sum_{j=1}^{k} j n_j = L - i - \ell.$$

The important point of the combinatorial method is that the generating function of paths with fixed sequence $(n_1, \dots, n_k)$, i.e., with fixed particle content may easily
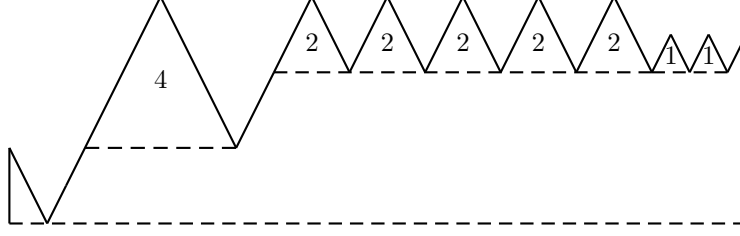
FIGURE 4. Minimal path corresponding to the path of Figure 3
with $k = 6$. All particles are moved to the right as much as possible
and arranged in decreasing order with respect to charge, such that
the maximal height of the path does not exceed k and such that
the final step remains upwards

be computed, and as a special case we quote [31, Prop. 3]

$$(3.17) \qquad G_{0,\ell}(L; n_1, \ldots, n_k; q) = q^{\frac{1}{4}mCm} \prod_{j=1}^{k-1} \begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix},$$

where the m_j follow from

$$(3.18) \qquad m = C^{-1}\Big[Le_1 + e_{k-\ell} - 2n\Big]$$

and where the n_j are subject to

$$(3.19) \qquad 2\sum_{j=1}^{k} jn_j = L - \ell.$$

The term $\frac{1}{4}mCm$ in the above summand is the weight of the 'minimal path of
content $(n_1, \ldots, n_k)$' corresponding to the special type of path shown in Figure 4.
All other paths with the same content have a weight of the form $\frac{1}{4}mCm + N$ where
N is a positive integer, and together they generate the product of q-binomials
in (3.17). The value of N for a non-minimal path P is given by the number of
'elementary' moves required to obtain P from its corresponding minimal path $P_{\min}$,
and, importantly, *only* depends on the shape of a path. We refer the reader to
[30, 31] for the precise details.

Let us now return to the problem of computing the generating function $G_\ell(L, M; q)$.
Since we can again make a particle decomposition of admissible paths, we may first
try to compute $G_\ell(L, M; n_1, \ldots, n_k; q)$. This follows from $G_{0,\ell}(L; n_1, \ldots, n_k; q)$ by
carrying out the following two transformations, the first acting on the generating
function and the second acting on the actual paths.

(1) Replace L by $L + M$ in $G_{0,\ell}(L; n_1, \ldots, n_k; q)$, so that we are counting paths
   from $(0, 0)$ to $(L + M, \ell)$, $(L + M + 1, \ell + 1)$.
(2) Translate each path counted by $G_{0,\ell}(L + M; n_1, \ldots, n_k; q)$ exactly L units
   to the left.

The first transformation simply means that (3.18) and (3.19) need to be replaced by (3.15b) and

$$(3.20) \qquad 2\sum_{j=1}^{k} jn_j = L + M - \ell.$$

The second transformation has a more subtle effect. The weight of a path is computed by summing up the x coordinates of $(+,+)$ and $(-,-)$ sequences of steps, and it is non-trivial to relate the weight of a left-translated path to its weight before translation. However, the crucial observation is that the weights of all paths with the same content $(n_1,\dots,n_k)$ are rescaled identically. Specifically, if the weight of the minimal path changes as

$$K \to K + \Delta K$$

as a result of the translation, then a non-minimal path of the same particle content changes as

$$K + N \to K + N + \Delta K.$$

This follows from the previously-mentioned facts that (i) the weight of a non-minimal path P is obtained from its minimal path $P_{\min}$ by counting the number, N, of elementary moves required to obtain P from $P_{\min}$, and (ii) N only depends on the shape of P, and is thus invariant under translations.

So all we have to do is (re)calculate the weight of the minimal path of content $(n_1,\dots,n_k)$ after its translation to the left. This is given by E with

$$E = \sum_{j=1}^{k} j(j-1)n_j^2 + 2\sum_{j=1}^{k}\sum_{l=j+1}^{k}(j-1)ln_jn_l + \ell\sum_{j=1}^{k}(j-1)n_j$$
$$+ \sum_{j=k-\ell+1}^{k}(\ell-k+j)n_j + \frac{1}{4}\ell(\ell+1) - L\left[\sum_{j=1}^{k}(j-1)n_j - \frac{1}{2}\ell\right].$$

When $L = 0$ (no left-translation) this is the weight of the minimal paths computed in [31, Eq. (3.13)]. Eliminating n_k using the restriction (3.20), and then writing the result in terms of the m_j instead of n_j using (3.15b), yields

$$E = \frac{1}{4}mCm - \frac{1}{2}Lm_1.$$

We therefore conclude that

$$(3.21) \qquad G_\ell(L,M;n_1,\dots,n_k;q) = q^{\frac{1}{4}mCm - \frac{1}{2}Lm_1}\prod_{j=1}^{k-1}\begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix},$$

where the m_j follow from (3.15b) and where the n_j must satisfy (3.20). To obtain the full generating function all we need to do is sum over all admissible sequences $(n_1,\dots,n_k)$. Now (3.20) may be rewritten as

$$\frac{L+M-\ell}{2k} + (C^{-1}n)_1 = n_1 + n_2 + \cdots + n_k$$

with $n = (n_1,\dots,n_{k-1})$. Since the right-hand side of (3.21) has no explicit dependence on n_k, summing over admissible sequences simply corresponds to summing over $n \in \mathbb{Z}^{k-1}$ such that

$$\frac{L+M-\ell}{2k} + (C^{-1}n)_1 \in \mathbb{Z}.$$

In conclusion, $G_\ell(L, M; q)$ is given by the right-hand side of (3.15a) completing the proof of (3.15).

Finally we need to return to (3.7b) with $\ell \in \{1, \ldots, k\}$. Using (3.10), (3.14) and

$$X_{r,s}^{(p,p')}(L, b; q) = X_{p-r,p'-s}^{(p,p')}(L, p' - b; q)$$

on the left-hand side and eliminating n in the exponent of q on the right-hand side we find the following variation on (3.15a):

(3.22)

$$\sum_{\substack{i=0 \\ i \equiv L \pmod 2}}^{k} q^{(\ell-1)(\ell-2i)/4} X_{k-\ell+1,k-i+1}^{(k+1,k+2)}(M, k-\ell+1; q) X_{k,k-i+1}^{(k+1,k+2)}(L, k+1; q^{-1})$$

$$= \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{L+M+\ell}{2k} + (C^{-1}n)_1 \in \mathbb{Z}}} q^{\frac{1}{4}mCm - \frac{1}{2}Lm_1} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix},$$

subject to (3.7a) and (3.7c).

This time both sides of (3.22) may be interpreted as the generating function $H_\ell(L, M; q)$ of lattice paths confined to the strip with weight function as before, but with initial vertex $(-L, k)$ and final two vertices $(M, k-\ell)$, $(M+1, k-\ell+1)$. Obviously there are no admissible paths unless (3.7a) and (3.7c) are satisfied.

Previously we defined the generating function $G_{i,\ell}(L; q)$ of paths in the strip with initial vertex $(0, i)$ and final two vertices (L, ℓ), $(L+1, \ell+1)$. Let us now also define $H_{i,\ell}(L; q)$ as the generating function of paths in the strip with initial vertex $(0, i)$ and final two vertices (L, ℓ), $(L+1, \ell-1)$.

Again a path P counted by $H_\ell(L, M; q)$ may be seen as the concatenation of a left-pointing path P_- from $(0, k-i)$ to $(-L, k)$ and a right-pointing path P_+ from $(0, k-i)$ to $(M, k-\ell)$, $(M+1, k-\ell+1)$, where $k \in \{0, \ldots, \ell\}$ and $i \equiv L \pmod 2$. This time we may add a down step from $(-L, k)$ to $(-L-1, k-1)$ to P_- without changing its weight (so that P_- terminates with a $(+, -)$ pair of steps). Reflecting P_- in the y-axis we thus find

$$H_\ell(L, M; q) = \sum_{\substack{i=0 \\ i \equiv L \pmod 2}}^{k} G_{k-i,k-\ell}(M; q) H_{k-i,k}(L; q^{-1}).$$

This includes the trivial relations

$$H_\ell(0, M; q) = G_{k,k-\ell}(M; q) \quad \text{and} \quad H_\ell(L, 0; q) = H_{k-\ell,k}(L; q^{-1}).$$

Once more using (3.16) as well as [2]

$$H_{i,\ell}(L; q) = q^{(i-\ell)(i-\ell+1)/4} X_{\ell,i+1}^{(k+1,k+2)}(L, \ell+1; q),$$

yields

$$H_\ell(L, M; q)$$

$$= \sum_{\substack{i=0 \\ i \equiv L \pmod 2}}^{k} q^{(\ell-1)(\ell-2i)/4} X_{k-\ell+1,k-i+1}^{(k+1,k+2)}(M, k-\ell+1; q) X_{k,k-i+1}^{(k+1,k+2)}(L, k+1; q^{-1}),$$

in accordance with the left-hand side of (3.22).

To obtain a fermionic evaluation of $H_\ell(L, M; q)$ we need [31, Prop. 3]

$$G_{k,k-\ell}(L; n_1, \ldots, n_k; q) = q^{\frac{1}{4}mCm} \prod_{j=1}^{k-1} \begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix},$$

where the m_j follow from

(3.23) $$m = C^{-1}\left[Le_1 + e_\ell - 2n\right]$$

and the n_j are constraint by

(3.24) $$2\sum_{j=1}^{k} jn_j = L + \ell - 2k.$$

Again the $\frac{1}{4}mCm$ in the above summand is the weight of the 'minimal path of content $(n_1, \ldots, n_k)$'.

To obtain $H_\ell(L, M; q)$ we again replace $L \to L + M$ in the expression for $G_{k,k-\ell}(L; n_1, \ldots, n_k; q)$ and then translate all paths by L units to the left. As before, the weight E of the minimal path of content $(n_1, \ldots, n_k)$ needs to be recalculated, yielding

$$E = \sum_{j=1}^{k} j(j-1)n_j^2 + 2\sum_{j=1}^{k}\sum_{l=j+1}^{k}(j-1)ln_l + (2k-\ell)\sum_{j=1}^{k}(j-1)n_j$$

$$+ \sum_{j=\ell+1}^{k}(j-\ell)n_j + \frac{1}{4}k(k-1) + \frac{1}{4}(k-\ell)(3k-\ell+1)$$

$$- L\left[\sum_{j=1}^{k}(j-1)n_j - \frac{1}{2}(\ell+1)\right].$$

By (3.23) and (3.24) with $L \to L + M$ this yields

$$E = \frac{1}{4}mCm - \frac{1}{2}Lm_1,$$

so that

(3.25) $$H_\ell(L, M; n_1, \ldots, n_k; q) = q^{\frac{1}{4}mCm - \frac{1}{2}Lm_1} \prod_{j=1}^{k-1} \begin{bmatrix} n_j + m_j \\ n_j \end{bmatrix},$$

where the m_j follow from (3.7c) and where the n_j must satisfy

(3.26) $$2\sum_{j=1}^{k} jn_j = L + M + \ell - 2k.$$

Once again, to obtain the full generating function we need to sum over all admissible sequences $(n_1, \ldots, n_k)$. Since (3.25) has not explicit n_k dependence and since (3.26) may be rewritten as

$$\frac{L + M + \ell}{2k} + (C^{-1}n)_1 = 1 + n_1 + n_2 + \cdots + n_k$$

with $n = (n_1, \ldots, n_{k-1})$, this results in the right-hand side of (3.22).

3.5. **Proof of Proposition 3.3.** Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ with finitely many λ_i unequal to zero. The nonzero λ_i are called the parts of λ. The weight $|\lambda|$ of λ is the sum of its parts. If $m_i(\lambda) = m_i$ is the multiplicity of the part i in λ we will also write $\lambda = (1^{m_1}, 2^{m_2}, \dots)$. Given a partition λ with largest part at most $k - 1$ we define

$$e_\lambda = e_{\lambda_1} + e_{\lambda_2} + \cdots = \sum_{i=1}^{k-1} m_i(\lambda) e_i.$$

For r an integer, $\sigma \in \{0, 1\}$ and λ a partition with largest part at most $k - 1$ the following identity was proved in [28, Cor. 4.1]:

(3.27a) $$\sum_{L=0}^{\infty} \frac{q^{L(L+r)/k}}{(q;q)_L (q;q)_{L+r}} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{2L+r-|\lambda|+k\sigma}{2k} - (C^{-1}n)_1 \in \mathbb{Z}}} q^{nC^{-1}(n-e_\lambda)} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}$$

$$= \frac{1}{(q;q)_\infty} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{r-|\lambda|+k\sigma}{2k} - (C^{-1}n)_1 \in \mathbb{Z}}} \frac{q^{nC^{-1}(n-e_\lambda)}}{(q;q)_{n_1} \cdots (q;q)_{n_{k-1}}},$$

where

(3.27b) $$m = C^{-1}\Big[(2L+r)e_{k-1} + e_\lambda - 2n\Big]$$

and

(3.27c) $$r + |\lambda| + \sigma k \equiv 0 \pmod 2.$$

Note that σ is fixed if k is odd, but can be either 0 or 1 when k is even.

If we choose λ and σ as

$$\lambda = (k - \ell, (k-1)^{n_0}) \qquad \text{and} \qquad \sigma \equiv n_0 + 1 \pmod 2$$

with $\ell \in \{1, \dots, k\}$ (when $\ell = 1$ or $\ell = k$ the above should read $\lambda = ((k-1)^{n_0+1})$ or $\lambda = ((k-1)^{n_0})$, respectively), then (3.27) becomes

(3.28a)

$$\sum_{L=0}^{\infty} \frac{q^{L(L+r)/k}}{(q;q)_L (q;q)_{L+r}} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{2L+r+n_0+\ell}{2k} - (C^{-1}n)_1 \in \mathbb{Z}}} q^{nC^{-1}(n - n_0 e_{k-1} - e_{k-\ell})} \prod_{i=1}^{k-1} \begin{bmatrix} n_i + m_i \\ n_i \end{bmatrix}$$

$$= \frac{1}{(q;q)_\infty} \sum_{\substack{n \in \mathbb{Z}^{k-1} \\ \frac{r+n_0+\ell}{2k} - (C^{-1}n)_1 \in \mathbb{Z}}} \frac{q^{nC^{-1}(n - n_0 e_{k-1} - e_{k-\ell})}}{(q;q)_{n_1} \cdots (q;q)_{n_{k-1}}},$$

where

(3.28b) $$m = C^{-1}\Big[(2L+r+n_0)e_{k-1} + e_{k-\ell} - 2n\Big]$$

and

(3.28c) $$n_0 + r + \ell \equiv 0 \pmod 2.$$

So far we have assumed that $\ell \in \{1, \dots, k\}$. If in (3.27) we choose

$$\lambda = ((k-1)^{n_0}) \qquad \text{and} \qquad \sigma \equiv n_0 \pmod 2$$

it easily follows that we obtain the $\ell = 0$ instance of (3.28).

The final step in our proof consists of changing the summation indices n_i to n_{k-i} on both sides of (3.28a) (on the left-hand side we of course also change $m_i \to m_{k-i}$). Since

$$(C^{-1}n)_1 + (C^{-1}n)_{k-1} \in \mathbb{Z}$$

this yields Proposition 3.3.

4. B = F IN THE LARGE k LIMIT

In the large k limit our proof that $B = S = F$ can be considerably simplified and made purely analytic, as will be demonstrated below.

In fact, since the equivalence between the spinon and fermionic forms of the graded characters rests on Proposition 3.3 which does not significantly simplify in the large k limit, we will only consider the $B = F$ correspondence here.

When k tends to infinity the bosonic character $\chi^{B}_{\ell,r}(q)$ of (1.18) trivializes to

$$V_r(q) - V_{r+2\ell+1}(q)$$

since only the singular vector of lowest conformal dimension yields a non-vanishing contribution. This should equate with the large k limit of the fermionic character $\chi^{F}_{\ell,r}(q)$ of (1.19).

Theorem 4.1. *Let ℓ to be a positive integer and r be any integer. Then*

$$(4.1) \quad \lim_{k \to \infty} \sum_{\substack{n_0,\dots,n_{k-1}=0 \\ \frac{n_0-r}{2k}+(C^{-1}n)_1 \in \mathbb{Z}}}^{\infty} \frac{q^{(r-n_0)(r+n_0+2\ell)/(4k)+\binom{n_0+1}{2}+nC^{-1}(n-n_0 e_1 - e_\ell)}}{(q;q)_{n_0} \cdots (q;q)_{n_{k-1}}}$$

$$= V_r(q) - V_{r+2\ell+1}(q).$$

We note that for later convenience we have used that $\chi_{\ell,r}(q) = \chi_{\ell,-r-2\ell}(q)$ on the left-hand side. Hence the restriction on the sum does not exactly match that of (3.2). We also remark that the summand on the left only makes sense for $k \geq \ell$. Since we consider the large k limit for fixed ℓ this is of course not a significant issue.

Before proving the above theorem we first introduce some more notation relating to partitions (see Section 3.5). The length $l(\lambda)$ of a partition λ is the number of parts (non-zero λ_i). If the weight of λ is n, i.e., $|\lambda| = n$ we write $\lambda \vdash n$ and say that λ is a partition of n. We identify a partition with its Ferrers graph, defined by the set of points in $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. The conjugate λ' of λ is the partition obtained by reflecting the diagram of λ in the main diagonal, so that, in particular, $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$.

Given a partition we set

$$n(\lambda) = \sum_{i \geq 1}(i-1)\lambda_i = \sum_{i \geq 1}\binom{\lambda'_i}{2}$$

and

$$b_\lambda(q) = \prod_{i \geq 1}(q;q)_{m_i(\lambda)} = \prod_{i \geq 1}(q;q)_{\lambda'_i - \lambda'_{i+1}}.$$

In order to take the limit in (4.1) we introduce two partitions λ and μ such that $l(\lambda) \leq \lfloor k/2 \rfloor$ and $l(\mu) \leq \lfloor (k-1)/2 \rfloor$ as follows

$$\lambda_i = n_i + n_{i+1} \cdots + n_{\lfloor k/2 \rfloor},$$
$$\mu_i = n_{k-i} + n_{k-i-1} \cdots + n_{1+\lfloor k/2 \rfloor}.$$

For example, when $k = 7$, $\lambda = (n_1 + n_2 + n_3, n_2 + n_3, n_3)$, $\mu = (n_4 + n_5 + n_6, n_4 + n_5, n_4)$ and conversely, $n = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3, \mu_3, \mu_2 - \mu_3, \mu_1 - \mu_2)$.

A simple calculation shows that for $\ell \in \{0, \ldots, \lfloor k/2 \rfloor\}$,

$$nC^{-1}(n - n_0 e_1 - e_\ell)$$
$$= \sum_{i \geq 1}(\lambda_i^2 + \mu_i^2) - n_0\lambda_1 - (\lambda_1 + \cdots + \lambda_\ell) - (|\lambda| + |\mu|)(|\lambda| + |\mu| + n_0 + \ell)/k$$
$$= 2n(\lambda') + 2n(\mu') + |\lambda| + |\mu| - n_0\lambda_1 - (\lambda_1 + \cdots + \lambda_\ell)$$
$$- (|\lambda| + |\mu|)(|\lambda| + |\mu| + n_0 + \ell)/k.$$

Also,

$$(C^{-1}n)_1 = \frac{1}{k}(|\mu| - |\lambda|) + \lambda_1.$$

Using these two results the large k limit can easily be taken leading to

$$\text{LHS}(4.1) = \sum_{\substack{n_0=0 \\ n_0 \equiv r \pmod 2}} \sum_{\substack{\lambda,\mu \\ |\lambda|-|\mu|=\frac{1}{2}(n_0-r)}} \times \frac{q^{\binom{n_0+1}{2}-n_0\lambda_1+2n(\lambda')+2n(\mu')+|\lambda|+|\mu|-(\lambda_1+\cdots+\lambda_\ell)}}{(q;q)_{n_0} \, b_{\lambda'}(q) b_{\mu'}(q)}.$$

To further simplify this we invoke Hall's identity [16]

$$(4.2) \qquad \sum_{\lambda \vdash j} \frac{q^{2n(\lambda)}}{b_\lambda(q)} = \frac{1}{(q;q)_j}$$

(with $\lambda \to \mu'$) to find

$$\text{LHS}(4.1) = \sum_{j=0}^{\infty} \sum_{\substack{m=0 \\ m \equiv r \pmod 2}}^{\infty} \sum_{\lambda \vdash j+\frac{1}{2}(m-r)} \frac{q^{j+\binom{m+1}{2}-m\lambda_1+2n(\lambda')+|\lambda|-(\lambda_1+\cdots+\lambda_\ell)}}{(q;q)_j (q;q)_m b_{\lambda'}(q)},$$

where we have also replaced n_0 by m. Key to showing that this is equal to the right-hand side of (4.1) is the identity

$$(4.3) \qquad \sum_{\substack{m=0 \\ m \equiv j \pmod 2}}^{\infty} \sum_{\lambda \vdash \frac{1}{2}(j+m)} \frac{q^{\binom{m+1}{2}-m\lambda_1+2n(\lambda')+|\lambda|-(\lambda_1+\cdots+\lambda_\ell)}}{(q;q)_m b_{\lambda'}(q)}$$
$$= \frac{1}{(q;q)_\infty} \sum_{i=0}^{\infty} \frac{q^{\binom{j-2i}{2}} - q^{\binom{j-2i-2\ell-1}{2}}}{(q;q)_i},$$

where j and ℓ are integers such that $\ell \geq 0$. Utilizing this with $j \to 2j - r$ leads to

$$\text{LHS}(4.1) = \frac{1}{(q;q)_\infty} \sum_{i,j=0}^{\infty} \frac{q^{\binom{2j-2i-r}{2}+j} - q^{\binom{2j-2i-2\ell-r-1}{2}+j}}{(q;q)_i (q;q)_j}$$

$$= V_r(q) - V_{r+2\ell+1}(q)$$

$$= \text{RHS}(4.1)$$

as desired.

The rest of this section is devoted to proving (4.3). First we change the summation index m to $2m - j$, cancel a common factor $q^{\binom{j}{2}}$ and use

$$\sum_{m=0}^{\infty} \sum_{\lambda \vdash m} f_{m,\lambda} \to \sum_{\lambda} f_{|\lambda|,\lambda}.$$

We are then left with the $a = q^{-j}$ instance of

$$(4.4) \quad \sum_\lambda \frac{a^{2|\lambda|-\lambda_1} q^{\binom{2|\lambda|+1}{2}-2|\lambda|\lambda_1+2n(\lambda')+|\lambda|-(\lambda_1+\cdots+\lambda_\ell)}}{(aq;q)_{2|\lambda|}\, b_{\lambda'}(q)}$$

$$= \frac{1}{(aq;q)_\infty} \sum_{i=0}^{\infty} \frac{a^{2i} q^{\binom{2i+1}{2}} - a^{2i+2\ell+1} q^{\binom{2i+2\ell+2}{2}}}{(q;q)_i}.$$

Our next step is to rename λ_1 as k and to let μ be the partition $\mu = (\lambda_2, \lambda_3, \dots)$. Also denoting $|\mu|$ by j this gives

$$\text{LHS}(4.4) = \sum_{j,k=0}^{\infty} \sum_{\mu \vdash j} \frac{a^{2j+k} q^{j(2j+2k+1)+k^2+k\delta_{\ell,0}+2n(\mu')+|\mu|-(\mu_1+\cdots+\mu_{\ell-1})}}{(aq;q)_{2j+2k}(q;q)_{k-\mu_1}\, b_{\mu'}(q)}$$

The sum over μ can now be performed by the following identity from [33]:

$$(4.5) \quad \sum_{\lambda \vdash j} \frac{q^{2n(\lambda')+|\lambda|-\sum_{i=1}^{m}\lambda_i}(q;q)_k}{(q;q)_{k-\lambda_1}\, b_{\lambda'}(q)} = \begin{bmatrix} k+j-1 \\ j \end{bmatrix} - (1-q^k)\begin{bmatrix} k+j-m-1 \\ j-m-1 \end{bmatrix},$$

for $m \in \{0, \dots, j\}$. We should remark that the above assumes a slightly different definition of the q-binomial coefficient than given in (3.1), namely

$$\begin{bmatrix} n+j \\ j \end{bmatrix} = \begin{cases} \dfrac{(q^{n+1};q)_j}{(q;q)_j} & j \in \{0,1,2,\dots\} \\ 0 & \text{otherwise.} \end{cases}$$

By (4.5) the μ-sum times $(q;q)_k$ gives

$$\begin{bmatrix} k+j-1 \\ j \end{bmatrix} - (1-q^k)\begin{bmatrix} k+j-\ell \\ j-\ell \end{bmatrix}$$

for $\ell > 0$ and

$$\begin{bmatrix} k+j-1 \\ j \end{bmatrix} - (1-q^k)\begin{bmatrix} k+j-1 \\ j-1 \end{bmatrix} = q^{-k}\begin{bmatrix} k+j-1 \\ j \end{bmatrix} - q^{-k}(1-q^k)\begin{bmatrix} k+j \\ j \end{bmatrix}$$

for $\ell = 0$. Combining the above two expressions we therefore find that

$$\text{LHS}(4.4) = \sum_{j,k=0}^{\infty} \frac{a^{2j+k} q^{j(2j+2k+1)+k^2}}{(aq;q)_{2j+2k}(q;q)_k} \left\{ \begin{bmatrix} k+j-1 \\ j \end{bmatrix} - (1-q^k)\begin{bmatrix} k+j-\ell \\ j-\ell \end{bmatrix} \right\}.$$

This should be equated with the right-hand side of (4.4). In fact, as we shall see, the following dissection takes place:

$$(4.6) \qquad \sum_{j,k=0}^{\infty} \frac{a^{2j+k}q^{j(2j+2k+1)+k^2}}{(aq;q)_{2j+2k}(q;q)_k} \begin{bmatrix} k+j-1 \\ j \end{bmatrix} = \frac{1}{(aq;q)_\infty} \sum_{i=0}^{\infty} \frac{a^{2i}q^{\binom{2i+1}{2}}}{(q;q)_i}$$

and

$$\sum_{j,k=0}^{\infty} \frac{a^{2j+k}q^{j(2j+2k+1)+k^2}}{(aq;q)_{2j+2k}(q;q)_{k-1}} \begin{bmatrix} k+j-\ell \\ j-\ell \end{bmatrix} = \frac{1}{(aq;q)_\infty} \sum_{i=0}^{\infty} \frac{a^{2i+2\ell+1}q^{\binom{2i+2\ell+2}{2}}}{(q;q)_i}.$$

Replacing $j \to j+\ell$, $k \to k+1$ and $a \to aq^{-2\ell-1}$ this last identity may also be stated as

$$(4.7) \qquad \sum_{j,k=0}^{\infty} \frac{a^{2j+k}q^{j(2j+2k+1)+k^2+k}}{(aq;q)_{2j+2k+1}(q;q)_k} \begin{bmatrix} k+j+1 \\ j \end{bmatrix} = \frac{1}{(aq;q)_\infty} \sum_{i=0}^{\infty} \frac{a^{2i}q^{\binom{2i+1}{2}}}{(q;q)_i}$$

independent of ℓ. Both (4.6) and (4.7) are special cases of the more general

$$(4.8) \qquad \sum_{j,k=0}^{\infty} \frac{a^{2j}b^kq^{j(j+1)+(j+k)^2}}{(bq;q)_{2j+2k}(q;q)_k} \frac{(b^2q^k/a^2;q)_j}{(q;q)_j} = \frac{1}{(bq;q)_\infty} \sum_{i=0}^{\infty} \frac{a^{2i}q^{\binom{2i+1}{2}}}{(q;q)_i}.$$

To prove this we shift the summation index $k \to k-j$ and employ basic hypergeometric notation to get

$$\text{LHS}(4.8) = \sum_{k=0}^{\infty} \frac{b^kq^{k^2}}{(q;q)_k(bq;q)_{2k}} \, {}_2\phi_1 \begin{bmatrix} a^2q^{1-k}/b^2, q^{-k} \\ 0 \end{bmatrix} ; q, bq^{2k+1} \end{bmatrix}.$$

By Heine's transformation [15, Eq. (III.2)]

$$_2\phi_1(a,b;c;q,z) = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty} \, {}_2\phi_1(abz/c,b;bz;q,c/b)$$

this may be rewritten as

$$\text{LHS}(4.8) = \sum_{k=0}^{\infty} \frac{b^kq^{k^2}}{(q,bq;q)_k} \lim_{\gamma\to0} {}_2\phi_1 \begin{bmatrix} a^2q^2/b\gamma, q^{-k} \\ bq^{k+1} \end{bmatrix} ; q, \gamma q^k \end{bmatrix}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{a^{2j}b^{k-j}q^{j^2+j+k^2}}{(q;q)_j(q;q)_{k-j}(bq;q)_{j+k}}.$$

Changing the order of the two sums and shifting $k \to k+j$ this becomes

$$\text{LHS}(4.8) = \sum_{j=0}^{\infty} \frac{a^{2j}q^{2j^2+j}}{(q;q)_j(bq;q)_{2j}} \lim_{\gamma,\delta\to\infty} {}_2\phi_1 \begin{bmatrix} \gamma, \delta \\ bq^{2j+1} \end{bmatrix} ; q, \frac{bq^{2j+1}}{\gamma\delta} \end{bmatrix}$$

$$= \frac{1}{(bq;q)_\infty} \sum_{j=0}^{\infty} \frac{a^{2j}q^{2j^2+j}}{(q;q)_j}$$

in accordance with the right-hand side of (4.8). To obtain the final expression on the right we have employed the q-Gauss sum [15, Eq. (II.8)]

$$_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}.$$

APPENDIX A. THE IDENTITY (1.12)

Up to trivial manipulations, the difference between (1.5) and (1.10) is due to a different representation of the Verma character $\hat{V}_t(q)$.

Indeed, we note that (1.9) can be written in q-hypergeometric notation as

$$\hat{V}_t(q) = \frac{q^{\max\{0,t\}}}{(q;q)_{|t|}} \, {}_2\phi_1(0,0;q^{|t|+1};q,q)$$

By Heine's transformation [15, Eq. (III.1)]

$$\,{}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \, {}_2\phi_1(c/b,z;az;q,b)$$

this yields

$$\hat{V}_t(q) = \frac{q^{\max\{0,t\}}}{(q;q)_\infty^2} \sum_{i=0}^\infty (-1)^i q^{\binom{i+1}{2}+i|t|}.$$

Since

$$(A.1) \qquad \sum_{i=0}^\infty (-1)^i q^{\binom{i+1}{2}+it} = 0$$

for $t \in \mathbb{Z}$ this may be simplified to

$$(A.2) \qquad \hat{V}_t(q) = \frac{1}{(q;q)_\infty^2} \sum_{i=0}^\infty (-1)^i q^{\binom{i+1}{2}-it}.$$

It is this 'bosonic' form for the Verma character that is responsible for those minus signs in (1.5) that are not due to the usual addition and subtraction of singular vectors.

Substituting (A.2) in (1.10) and using that $2r = m - \ell$ we obtain

$$(A.3) \quad \hat{\chi}_{\ell,r}(q) = \frac{1}{\eta^2(q)} \sum_{j=-\infty}^\infty \sum_{i=0}^\infty (-1)^i q^{-(m+ik)^2/4k}$$

$$\times \left\{ q^{\left(\ell+1+(i+2j)(k+2)\right)^2/4(k+2)} - q^{\left(\ell+1-(i-2j)(k+2)\right)^2/4(k+2)} \right\}.$$

Replacing $j \to -j$ in the sum corresponding to the second term in the summand and then splitting the sum over j and using (A.1) in the form $\sum_{i\geq 0} \cdots = -\sum_{i<0} \cdots$ completes the proof of (1.12).

We should remark that a similar kind of rewriting of the graded verma character (1.17) may be carried out to yield the bosonic form

$$V_t(q) = \frac{1}{(q;q)_\infty^3} \sum_{j=-\infty}^\infty \sum_{i=0}^\infty (-1)^i q^{\binom{2j-t}{2}+\binom{i+1}{2}+j(i+1)}.$$

An alternative — albeit somewhat indirect — demonstration of (1.12) is to show that (1.10) follows from (1.2) and (1.3). This is easily achieved as follows. By the Jacobi triple product identity (2.16) and the expansion (2.15) the denominator of (1.4) may be put as

$$\frac{1}{\displaystyle\sum_{\sigma\in\{\pm 1\}} \Theta_\sigma^{(2)}(x;q)} = \frac{q^{-1/12}x^{1/2}}{\eta(q)} \sum_{n=-\infty}^\infty x^{-n}\hat{V}_n(q),$$

with $\hat{V}_n(q)$ the character (1.9).

Multiplying the above expression by the numerator of (1.4) yields

$$\hat{\chi}_\ell(x;q) = \frac{q^{-1/12}}{\eta(q)} \sum_{\sigma \in \{\pm 1\}} \sigma$$
$$\times \sum_{j,n=-\infty}^{\infty} x^{-n-j(k+2)-\sigma(\ell+1)/2+1/2} q^{(k+2)\left(\sigma j + \frac{\ell+1}{2(k+2)}\right)^2} \hat{V}_n(q).$$

Shifting $n \to n - (k+2)j - (\sigma-1)(\ell+1)/2$ and replacing j by σj leads to

$$\hat{\chi}_\ell(x;q) = \frac{1}{\eta(q)} \sum_{n=-\infty}^{\infty} x^{-n-\ell/2} q^{(n+\ell/2)^2/k} \hat{\chi}_{\ell,n}^{B'}(q)$$

with $\hat{\chi}_{\ell,r}^{B'}(q)$ given by (1.10). Finally setting $n = kj + (m-\ell)/2$ with $1-k \le m \le k$ and using the symmetry $\hat{\chi}_{\ell,r}(q) = \hat{\chi}_{\ell,r+jk}(q)$ yields

$$\hat{\chi}_\ell(x;q) = \sum_{\substack{m=1-k \\ m-\ell \text{ even}}}^{k} \hat{\chi}_{\ell,\frac{m-\ell}{2}}^{B'}(q) K_m(x;q).$$

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